Functional Analysis - Summary

1. Normed spaces

1. Triangle inequalities for $\mathbb{R}$: $|x + y| \leq |x| + |y|$, \( \sum_{i=1}^{N} a_i \leq \sum_{i=1}^{N} |a_i| \).

2. Cauchy’s inequality: For $a_i, b_i \in \mathbb{R}$,
\[
\sum_{i=1}^{N} |a_i b_i| \leq \left( \sum_{i=1}^{N} |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^{N} |b_i|^2 \right)^{1/2}.
\]

3. Hölder’s inequality: For $p, q > 1$, with $1/p + 1/q = 1$,
\[
\sum_{i=1}^{N} |a_i b_i| \leq \left( \sum_{i=1}^{N} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{N} |b_i|^q \right)^{1/q}.
\]

4. Minkowski’s inequality: For for $p > 1$:
\[
\left( \sum_{i=1}^{N} |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{N} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{N} |b_i|^p \right)^{1/p}.
\]

5. Integral versions of the above, eg for Cauchy’s inequality:
\[
\int |fg| \leq \left( \int f^2 \right)^{1/2} \left( \int g^2 \right)^{1/2}.
\]

6. A normed space $(X, \|\|)$ is a (real or complex) vector space $X$ together with a norm $\|\|$, i.e. a function $\|\| : X \to \mathbb{R}$, such that
(i) $\|x\| \geq 0$ with $\|x\| = 0$ if and only if $x = 0$ (positivity),
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and scalars $\lambda$ (scalar property).
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality).

7. In a normed space:
\[
\|x_1 + \ldots + x_n\| \leq \|x_1\| + \ldots + \|x_n\|,
\]
\[
\|\|x\| - \|y\|\| \leq \|x - y\| \quad \text{(reverse triangle)}.
\]

8. $\text{dist}(x, y) = \|x - y\|$ defines a metric on $X$, i.e. (i) $\text{dist}(x, y) \geq 0$ with equality iff $x = y$, (ii) $\text{dist}(x, y) = \text{dist}(y, x)$, (iii) $\text{dist}(x, y) \leq \text{dist}(x, z) + \text{dist}(z, y)$.

9. Examples of normed spaces:
- $\mathbb{R}^N = \{x = (x_1, \ldots, x_N)\}$ with $\|x\|_\infty = \max_{1 \leq i \leq N} |x_i|$, or $\|x\|_1 = \sum_{i=1}^{N} |x_i|$, or $\|x\|_2 = (\sum_{i=1}^{N} |x_i|^2)^{1/2}$, or for any $p \geq 1$, $\|x\|_p = (\sum_{i=1}^{N} |x_i|^p)^{1/p}$.
- $C[0, 1] = \{\text{continuous functions on } [0, 1]\}$ with $\|f\|_\infty = \sup_{0 \leq t \leq 1} |f(t)|$, or $\|f\|_1 = \int_0^1 |f(t)| dt$, or $\|f\|_2 = (\int_0^1 (f(t)^2 dt)^{1/2}$. 

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2. Completeness and the contraction mapping theorem

1. A sequence \((x_n)\) in a normed space \((X,\|\|)\) is convergent to \(x \in X\) if given \(\epsilon > 0\) there exists \(n_0\) such that \(\|x_n - x\| < \epsilon\) for all \(n \geq n_0\). Equivalently if \(\|x_n - x\| \to 0\) as \(n \to \infty\).

2. A sequence \((x_n)\) in a normed space \((X,\|\|)\) is a Cauchy sequence if given \(\epsilon > 0\) there exists \(n_0\) such that \(\|x_n - x_m\| < \epsilon\) for all \(n, m \geq n_0\).

3. In a normed space every convergent sequence is a Cauchy sequence.

4. A normed space in which every Cauchy sequence is convergent (to a point in the space) is called a complete normed space or a Banach space. Similarly, a subset \(Y\) of a normed space in which every Cauchy sequence is convergent is called complete.

5. Every bounded sequence in \(\mathbb{R}\) has a convergent subsequence.

6. The following normed spaces are complete:
   - \((\mathbb{R}, \|\|)\) (This is just the general principle of convergence).
   - \((\mathbb{R}^N, \|\|_\infty)\), \((\mathbb{R}^N, \|\|_1)\), \((\mathbb{R}^N, \|\|_2)\).
   - \((C[0,1], \|\|_\infty)\).
   - \((L^1[0,1], \|\|_1)\), \((L^2[0,1], \|\|_2)\), \((L^p[0,1], \|\|_p)\) for all \(p \geq 1\).

7. Let \(Y\) be a closed subspace or subset of a Banach space \((X,\|\|)\). Then \(Y\) is complete.
8 Contraction mapping theorem: Let $X$ be a Banach space or a closed subset of a Banach space with norm $\| \cdot \|$. Let $T : X \to X$ be a contraction, i.e. there is a constant $0 < k < 1$ such that

$$\| T(x) - T(y) \| \leq k \| x - y \| \quad (x, y \in X).$$

Then $T$ has a unique fixed point, i.e. there is a unique $x \in X$ such that $T(x) = x$. Moreover, for all $y \in X$, we have $T^n(y) \to x$.

9 To apply the contraction mapping theorem, write the problem in the form $Tx = x$ and show $T$ is a contraction on some complete set, to deduce that the problem has a unique solution. $T$ might be a mapping on $\mathbb{R}$ (for solution of equations of one variable) or on $\mathbb{R}^N$ (for solution of equations in $N$ variables) or an integral operator (for solution of differential equations).

3. Finite dimensional normed spaces

1 Every bounded sequence in $(\mathbb{R}^N, \| \cdot \|_\infty)$ has a convergent subsequence.

2 Two norms $\| \cdot \|_A$ and $\| \cdot \|_B$ on a vector space $X$ are equivalent if there are $a, b > 0$ such that

$$a \| x \|_B \leq \| x \|_A \leq b \| x \|_B \quad \text{for all } x \in X.$$

3 If $\| \cdot \|_A$ and $\| \cdot \|_B$ are equivalent then $x_n \to x$ in $\| \cdot \|_A$ if and only if $x_n \to x$ in $\| \cdot \|_B$. Similarly, the Cauchy sequences, limit points, closed sets, etc. are the same in both norms.

4 If $X$ is a finite dimensional vector space then all norms on $X$ are equivalent.

5 In a finite dimensional vector space, the convergent sequences, Cauchy sequences, limit points, closed sets, etc. are independent of the norm used.

4. The space $(C[0, 1], \| \cdot \|_\infty)$

The material of this chapter is stated for $C[0, 1]$, but applies to continuous functions with the supremum norm on any “compact subset of a metric or topological space”. In particular $[0, 1]$ may be replaced with any bounded closed interval $[a, b]$, the interval $[0, 2\pi]$ with 0 and $2\pi$ identified, or the square $[0, 1] \times [0, 1]$.

1 A vector subspace $A$ of $C[0, 1]$ is an algebra if $fg \in A$ whenever $f \in A$ and $g \in A$. (e.g. $C[0, 1]$, polynomials, trigonometric polynomials, etc. are algebras).

2 Stone-Weierstrass Theorem: Let $A$ be a subalgebra of $(C[0, 1], \| \cdot \|_\infty)$ such that

(a) $A$ contains the constant functions, and

(b) $A$ ‘separates the points of $[0, 1]$’, i.e. if $t, u \in [0, 1]$ then we may find $f \in A$ such that $f(t) \neq f(u)$.

Then $A$ is dense in $C[0, 1]$, that is $\overline{A} = C[0, 1]$. 

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3 Weierstrass Approximation Theorem: The polynomials on \([0, 1]\) are dense in \((C[0, 1], \| \cdot \|_\infty)\).
The infinitely differentiable functions on \([0, 1]\) are dense in \((C[0, 1], \| \cdot \|_\infty)\).
The two-variable polynomials \(p(x, y)\) on \([0, 1] \times [0, 1]\) are dense in \((C([0, 1] \times [0, 1]), \| \cdot \|_\infty)\).
The trigonometric polynomials on \([0, 2\pi]\) with 0 and 2\pi identified, i.e. functions of the form \(\sum_{j=1}^{n} a_j \sin j\tau + \sum_{j=0}^{m} b_j \cos j\tau\), are dense in \((C[0, 2\pi], \| \cdot \|_\infty)\).

4 To prove a result that applies to all continuous functions in, say, \(C[0, 1]\), it is often easier to prove it for a dense subset of nice functions such as the polynomials and use density to transfer the result to general continuous functions in \(C[0, 1]\).

5. Operators on normed spaces

1 For \(X, Y\) normed spaces, we call a mapping \(T : X \to Y\) an operator. \(T\) is called linear if

\[
T(\lambda x + \mu x') = \lambda T(x) + \mu T(x') \quad \text{(for all } x, x' \in X \text{ and scalars } \lambda, \mu),
\]

Equivalently \(T\) is linear if both \(T(x + x') = T(x) + T(x')\) and \(T(\lambda x) = \lambda T(x)\).

\(T\) is continuous if \(T(x_n) \to T(x)\) whenever \(x_n \to x\).

\(T\) is bounded if there is a number \(c\) such that \(\|Tx\| \leq c \|x\|\) for all \(x \in X\).

2 Let \(T : X \to Y\) be linear. Then \(T\) is continuous if and only if \(T\) is bounded.

3 \(B(X, Y)\) denotes the set of bounded linear operators from \(X\) to \(Y\). (We abbreviate \(B(X, X)\) to \(B(X)\) for the operators from \(X\) to itself.)

4 For \(T \in B(X, Y)\) the induced norm \(\|T\|\) on \(T\) is given by

\[
\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|}{\|x\|} = \sup_{x : \|x\| = 1} \|Tx\|,
\]

that is the smallest number \(c\) such that \(\|Tx\| \leq c \|x\|\) for all \(x \in X\).

5 \(B(X, Y)\) is a vector space, which becomes a normed space under the induced norm. If \(Y\) is complete then so is \(B(X, Y)\).

6 If \(T \in B(X, Y)\) and \(U \in B(Y, Z)\) then \(UT \in B(X, Y)\) with \(\|UT\| \leq \|U\| \|T\|\). In particular if \(T \in B(X)\) then \(\|T^n\| \leq \|T\|^n\) for positive integers \(n\).

7 The identity operator \(I\) on \(X\) is defined by \(Ix = x\) for all \(x \in X\). Thus \(IT = TI = T\) if \(T \in B(X)\).

8 \(T \in B(X)\) is invertible if there is a bounded linear operator \(T^{-1} \in B(X)\) such that \(TT^{-1} = I = T^{-1}T\).

9 If \(T, U \in B(X)\) then \(TU\) is invertible if and only if \(T\) and \(U\) are both invertible, in which case \((TU)^{-1} = U^{-1}T^{-1}\).
Let $B$ be a Banach space and let $T \in B(X)$. If $\|T\| < 1$ then $T$ is invertible, with $(I - T)^{-1} = I + T + T^2 + \ldots$ (which is convergent in $B(X)$).

Let $B(X)$ be a complex Banach space and let $T \in B(X)$. The spectrum of $T$ is

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible} \}.$$

If $\lambda \in \mathbb{C}$ is such that $(T - \lambda I)x = 0$ for some $x \neq 0$ we call $\lambda$ an eigenvalue of $T$ and $x$ an eigenvector (or sometimes an eigenfunction). Eigenvalues all belong to the spectrum.

**Spectral mapping theorem:** Let $p$ be a polynomial. Then $\sigma(p(T)) = p(\sigma(T))$.

### 6. Inner product spaces and Hilbert space

An inner product space is a vector space $X$ over $\mathbb{R}$ or $\mathbb{C}$ (usually $\mathbb{C}$ in what follows) with an inner product $\langle x, y \rangle \in \mathbb{C}$ (or $\mathbb{R}$) such that:

(i) $\langle \lambda x + \mu y, w \rangle = \lambda \langle x, w \rangle + \mu \langle y, w \rangle$ (linearity),

(ii) $\langle x, y \rangle = \langle y, x \rangle$ (symmetry),

(iii) $\langle x, x \rangle$ is a real number, and $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ if and only if $x = 0$ (positivity).

From (i) and (ii) above $\langle w, \lambda x + \mu y \rangle = \lambda \langle w, x \rangle + \mu \langle w, y \rangle$ (that is the ‘dot product’) is a (real) Hilbert space; the norm is $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

$C[0, 1]$ with $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt$ is an inner product space but is NOT complete; the norm is $\|f\|_2 = \left( \int_0^1 |f(t)|^2 dt \right)^{1/2}$.

$L^2[0,1] = \{ f : f \text{ is square integrable on } [0,1] \text{ with } \int_0^1 |f(t)|^2 dt < \infty \}$ with $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt$ is a Hilbert space; the norm is $\|f\|_2 = \left( \int_0^1 |f(t)|^2 dt \right)^{1/2}$.
Let \( w(t) \) be a real ‘weight’ function. Then \( H = \{ f : \int_0^1 |f(t)|^2 w(t) dt < \infty \} \) with \( <f, g> = \int_0^1 f(t)g(t)w(t) dt \) is a Hilbert space; the norm is \( \|f\| = \left( \int_0^1 |f(t)|^2 w(t)^2 dt \right)^{1/2} \).

From now on let \( H \) be a Hilbert space.

5 \( x \) and \( y \) are orthogonal or perpendicular if \( <x, y> = 0 \).

6 A set of elements \( \{e_i\} \subset H \) is an orthonormal set if \( <e_i, e_j> = 0 \) for all \( i \neq j \) and \( \|e_i\| = <e_i, e_i> = 1 \) for all \( i \).

7 The span of a subset \( \{w_i\} \) of \( H \) is the vector subspace consisting of all finite linear combinations \( \sum_i a_i w_i \).

8 If \( \{e_i\} \) is orthonormal set with span\{e_i\} dense in \( H \), we call \( \{e_i\} \) a complete orthonormal system.

9 Schwartz’ inequality: In an inner product space
\[
|<x, y>| \leq \|x\|\|y\|.
\]

10 Parallelogram law: In an inner product space
\[
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.
\]

11 Let \( x \in H \) and let \( S \) be a dense subset of \( H \). If \( <x, s> = 0 \) for all \( s \in S \) then \( x = 0 \). In particular, if \( \{e_i\} \) is complete and \( <x, e_i> = 0 \) for all \( e_i \) then \( x = 0 \).

12 Let \( \{e_i\} \) be a complete orthonormal system (i.e with span\{e_i\} dense in \( H \)). Then if \( x \in H \),
\[
x = \sum_{i=1}^{\infty} <x, e_i> e_i \quad \text{(with this series convergent in } H),
\]
with
\[
\|x\| = \sum_{i=1}^{\infty} |<x, e_i>|^2 \quad \text{(Parseval’s inequality)}.
\]

13 Gram-Schmidt process: Let \( (x_1, x_2, \ldots) \) be linearly independent in a Hilbert space \( H \). Then we may find an orthonormal sequence \( (e_1, e_2, \ldots) \) such that for each \( n \) we have span\{\(x_1, x_2, \ldots\} = \text{span}\{e_1, e_2, \ldots\}. \) In fact we get \( e_n \) inductively by \( e_1 = x_1/\|x_1\| \) and
\[
e_{n+1} = \left( x_{n+1} - \sum_{i=1}^{n} <x_{n+1}, e_i> e_i \right) / \|x_{n+1} - \sum_{i=1}^{n} <x_{n+1}, e_i> e_i \|.
\]

14 An orthonormal set \( \{e_i\} \) is a basis for \( H \) if for every \( x \in H \) we have \( x = \sum_{i=1}^{\infty} a_i e_i \) for some \( a_i \) with the series convergent.

15 If for some sequence \( (x_i) \), span\{\(x_1, x_2, \ldots\} \) is dense in \( H \) then \( H \) has an orthonormal basis.
7. The spectral theorem

Throughout this section $H$ is an infinite dimensional complex Hilbert space (with a countable basis).

1. An operator $T \in \mathcal{B}(H)$ is self-adjoint or Hermitian if $<Tx, y> = <x, Ty>$ for all $x, y \in H$.

2. If $T$ is self-adjoint then $<Tx, x>$ is real.

3. If $T$ is self-adjoint then
   (i) the eigenvalues of $T$ are all real,
   (ii) eigenvectors corresponding to different eigenvalues are orthogonal.

4. Let $T$ be self-adjoint. Then $\|T\| = \sup_{\|x\|=1} |<Tx, x>|$.

5. An operator $T \in \mathcal{B}(H)$ is compact if for every bounded sequence $(x_n)$ the sequence $(Tx_n)$ has a convergent subsequence. (Think of a compact operator as one for which the image is ‘nearly finite dimensional’.)

6. Let $T : L^2[0, 1] \to L^2[0, 1]$ be such that the following ‘equicontinuity condition’ holds: for all $\epsilon > 0$ there exists $\delta > 0$ such that $|Tf(x) - Tf(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $f \in L^2[0, 1]$ such that $\|f\|_2 \leq 1$. Then $T$ is compact.

7. If $T$ is compact and $r > 0$ then there are at most finitely many eigenvalues (counted according to multiplicity) such that $|\lambda| \geq r$.

8. Let $T$ be a compact self-adjoint operator on $H$. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue of $T$.

9. Spectral theorem for compact self-adjoint operators: Let $T$ be a compact self-adjoint operator on $H$. Then there is an orthonormal sequence $(e_n)$ of eigenvectors in $H$, with $Te_n = \lambda_n e_n$, say, with $\lambda_n \neq 0$ and $\lambda_n \to 0$. Then if $x \in H$ we may write

   $$x = \sum_{n=1}^{\infty} a_n e_n + y,$$

   where $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, and $y \in H$ satisfies $<y, e_n> = 0$ for all $n$ and $Ty = 0$. In particular

   $$Tx = \sum_{n=1}^{\infty} a_n \lambda_n e_n.$$

10. Let $T$ be a compact self-adjoint operator on $H$. Then $H$ has an orthonormal basis of eigenvectors of $T$. 