

Exceptional sets for self-affine fractals

K.J. Falconer and J. Miao

*Mathematical Institute, University of St Andrews, North Haugh, St Andrews,
Fife, KY16 9SS, Scotland*

Abstract

Under certain conditions the ‘singular value function’ formula gives the Hausdorff dimension of self-affine fractals for almost all parameters in a family. We show that the size of the set of exceptional parameters is small both in the sense of Hausdorff dimension and Fourier dimension.

1 Introduction

Self-affine sets are the attractors of iterated function systems consisting of affine contractions and are of interest as linearisations of non-conformal hyperbolic systems, see [6]. However, the dimensional properties of self-affine sets are much more subtle than their self-similar counterparts.

For $N \geq 2$ let T_1, \dots, T_N be a set of linear contractions on \mathbb{R}^n , let $a_1, \dots, a_N \in \mathbb{R}^n$ be a set of translation vectors, and define affine transformations $S_1, \dots, S_N : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$S_i(x) = T_i(x) + a_i \quad (i = 1, \dots, N). \quad (1.1)$$

The set of contractions $\{S_1, \dots, S_N\}$ forms an *iterated function system* (IFS). By the well-known theorem of Hutchinson, see [7, 8], this IFS has a unique *attractor*, that is a unique non-empty compact $F(\mathbf{a}) \subset \mathbb{R}^n$ such that

$$F(\mathbf{a}) = \bigcup_{i=1}^N S_i(F(\mathbf{a})). \quad (1.2)$$

We write $F(\mathbf{a})$ for the attractor of (1.2) to emphasise its dependence on the vector of translations $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^{nN}$. Since the S_i are affine transformations, we refer to $F(\mathbf{a})$ as a *self-affine* set.

Determination of the Hausdorff and box dimensions of self-affine sets is a challenging problem, see [4, 5, 9, 13, 15] for various studies. It turns out that, given T_1, \dots, T_N , there is a number $d(T_1, \dots, T_N)$, see (2.4), such that

$$\dim_{\text{H}} F(\mathbf{a}) = \dim_{\text{B}} F(\mathbf{a}) = \min\{n, d(T_1, \dots, T_N)\} \quad (1.3)$$

for all almost all $\mathbf{a} \in \mathbb{R}^{nN}$ in the sense of nN -dimensional Lebesgue measure, where \dim_{H} and \dim_{B} denote Hausdorff and box dimension respectively. Formula (1.3) is valid

under various conditions on the $\|T_i\|$, in particular if $\|T_i\| < \frac{1}{2}$ for all i . (Note that $\min\{n, d(T_1, \dots, T_N)\}$ is always an upper bound for the dimensions.)

Whilst the set of ‘exceptional’ or ‘bad’ parameters \mathbf{a} for which (1.3) fails has nN -dimensional measure 0, this exceptional set may be quite small in the sense that its dimension may necessarily be rather less than nN . We define the *exceptional sets* of parameters

$$E(s) = \{\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^n : \dim_{\mathbb{H}} F(\mathbf{a}) < s\}$$

where $0 < s \leq \min\{n, d(T_1, \dots, T_N)\}$, and in this paper we obtain upper bounds for the dimension of $E(s)$. In Section 4 we use integral estimates to show that $\dim_{\mathbb{H}} E(s) \leq nN - c(d - s)$ where $d = d(T_1, \dots, T_N)$ for a suitable positive constant c . In section 5 we use Fourier transform methods to show that the Fourier dimension $\dim_{\mathbb{F}} E(s) \leq 2s$. In some ways the Fourier dimension bounds are more natural. Although many sets, particular randomly constructed sets, are Salem sets, that is have equal Hausdorff and Fourier dimensions, sets with inherently regular construction tend to have small Fourier dimension, for example the middle third Cantor set has Fourier dimension 0. Since self-affine sets with exceptionally small dimension occur when the iterated components resonate in some way, saying that the Fourier dimension of the exceptional set is small allows the possibility of it having rather larger Hausdorff dimension if it is of a ‘more regular’ construction.

Dimensions of exceptional sets have been examined in various other situations, for example for projections of a set onto subspaces [2, 11] and in a very general setting in [12].

2 Code space notation

It is standard in IFS theory to code points of the attractors and the sets and functions involved in their construction by sequences or words made up from the integers $\{1, 2, \dots, N\}$.

For each $k = 0, 1, 2, \dots$ let $\mathbf{J}_k = \{(i_1, \dots, i_k) : 1 \leq i_j \leq N\}$ be the set of sequences of length k , with \mathbf{J}_0 containing only the null sequence \emptyset . Let $\mathbf{J} = \bigcup_{k=0}^{\infty} \mathbf{J}_k$ be the set of all finite sequences, and let $\mathbf{J}_{\infty} = \{(i_1, i_2, \dots) : 1 \leq i_j \leq N\}$ be the corresponding set of infinite sequences. We abbreviate members of \mathbf{J} or \mathbf{J}_{∞} as $\mathbf{i} = (i_1, \dots, i_k)$, etc., and denote the number of terms in $\mathbf{i} \in \mathbf{J}$ by $|\mathbf{i}|$. If $\mathbf{i}, \mathbf{j} \in \mathbf{J}$ or if $\mathbf{i} \in \mathbf{J}$ and $\mathbf{j} \in \mathbf{J}_{\infty}$, we denote by \mathbf{ij} the sequence obtained by juxtaposition of the terms of \mathbf{i} and \mathbf{j} . If \mathbf{i} is a curtailment of \mathbf{j} , that is if $\mathbf{j} = \mathbf{ii}'$ for some \mathbf{i}' , we write $\mathbf{i} < \mathbf{j}$. If $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$, then $\mathbf{i} \wedge \mathbf{j} \in \mathbf{J}$ denotes the maximal common initial subsequence of both \mathbf{i} and \mathbf{j} .

We denote compositions of mappings indexed by \mathbf{J} by $T_{\mathbf{i}} = T_{i_1} T_{i_2} \cdots T_{i_k}$ where $\mathbf{i} = (i_1, i_2, \dots, i_k)$, etc.

We topologize \mathbf{J}_{∞} using the metric $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$ for distinct $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$ to make \mathbf{J}_{∞} into a compact metric space. We define the cylinder by $\mathcal{C}_{\mathbf{i}} = \{\mathbf{j} \in \mathbf{J}_{\infty} : \mathbf{i} < \mathbf{j}\}$ for $\mathbf{i} \in \mathbf{J}$; the set of cylinders $\{\mathcal{C}_{\mathbf{i}} : \mathbf{i} \in \mathbf{J}\}$ forms a base of open and closed neighborhoods for \mathbf{J}_{∞} .

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non-singular linear contraction. The *singular values* $\alpha_i \equiv \alpha_i(T)$ of T ($i = 1, \dots, n$) are the positive square roots of the eigenvalues of T^*T , where T^* is the transpose or adjoint of T . Equivalently they are the lengths of the principal semi-axes of the image $T(B)$ of the unit ball B . We adopt the convention that $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq$

$\alpha_n > 0$. The *singular value function* $\phi^s(T)$ is then defined for $0 \leq s \leq n$ as

$$\phi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \alpha_m^{s-m+1},$$

where m is the integer such that $m - 1 < s \leq m$, with the convention that $\phi^s(T) = (\alpha_1 \alpha_2 \cdots \alpha_n)^{s/n}$ if $s \geq n$.

The singular value function $\phi^s(T)$ is decreasing in s and is submultiplicative, that is $\phi^s(T_{\mathbf{ij}}) \leq \phi^s(T_{\mathbf{i}})\phi^s(T_{\mathbf{j}})$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{J}$. As a consequence, for each $s > 0$, the limit $\lim_{k \rightarrow \infty} [\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_{\mathbf{i}})]^{1/k}$ exists, and is continuous and strictly decreasing in s . Thus there is a unique $s > 0$ such that this limit equals 1, and we define

$$d(T_1, \dots, T_N) = \left\{ s : \lim_{k \rightarrow \infty} \left[\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_{\mathbf{i}}) \right]^{1/k} = 1 \right\}. \quad (2.4)$$

It turns out that, under certain conditions, for example if $\|T_i\| < \frac{1}{2}$ for almost all i , $\min\{n, d(T_1, \dots, T_N)\}$ equals the Hausdorff and box dimension of $F(\mathbf{a})$ for almost all $\mathbf{a} \in \mathbb{R}^{nN}$ in the sense of nN -dimensional Lebesgue measure, see [4, 5, 15].

We write

$$\alpha_+ = \max_{i=1, \dots, N} \alpha_1(T_i) \quad \text{and} \quad \alpha_- = \min_{i=1, \dots, N} \alpha_n(T_i).$$

This gives the bounds

$$\alpha_-^{s|\mathbf{i}|} \leq \phi^s(T_{\mathbf{i}}) \leq \alpha_+^{s|\mathbf{i}|} \quad (\mathbf{i} \in \mathbf{J}).$$

By the general IFS theory, since the S_1, \dots, S_N are contractions, for any non-empty compact set $K \subset \mathbb{R}^n$ with $S_i(K) \subset K$ for all $i = 1, \dots, N$, the attractor F is given by

$$F = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{i} \in \mathbf{J}_k} S_{i_1} S_{i_2} \cdots S_{i_k}(K),$$

where $\mathbf{i} = (i_1, \dots, i_k)$. Moreover, writing

$$\begin{aligned} x_{\mathbf{i}}(\mathbf{a}) &= \lim_{k \rightarrow \infty} (T_{i_1} + a_{i_1})(T_{i_2} + a_{i_2}) \cdots (T_{i_k} + a_{i_k})(0) \\ &= a_{i_1} + T_{i_1} a_{i_2} + T_{i_1} T_{i_2} a_{i_3} + \cdots \end{aligned} \quad (2.5)$$

for $\mathbf{i} = (i_1, i_2, \dots)$ and $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^{nN}$, the attractor is given by

$$F(\mathbf{a}) = \bigcup_{\mathbf{i} \in \mathbf{J}_{\infty}} x_{\mathbf{i}}(\mathbf{a}),$$

where the union is not necessarily disjoint.

3 Integral estimates

In this section we derive several integral estimates that we require. We denote the closed ball of radius r with center x by $B(x, r)$.

Lemma 3.1 *Let $0 < \rho_1 \leq \dots \leq \rho_n$ and let E be the parallelepiped $E = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_1| \leq \rho_1, \dots, |x_n| \leq \rho_n\}$. Let ν_0 be a Borel measure on \mathbb{R}^n satisfying $\nu_0(B(x, r)) \leq c_0 r^p$ for all $x \in \mathbb{R}^n$ and $r > 0$, where $0 \leq p \leq n$ and c_0 is a constant. Then*

$$\nu_0(E) \leq c_1 \rho_1 \cdots \rho_n \rho_1^{p-n}, \quad (3.6)$$

where c_1 depends only on n, p and c_0 .

Proof. Let $C(x, r)$ denote the closed hypercube of centre x and side r . For all $x \in \mathbb{R}^n$ we have $C(x, \rho_1) \subset B(x, \rho_1\sqrt{n}/2)$, so taking measures

$$\nu_0(C(x, \rho_1)) \leq \nu_0(B(x, \rho_1\sqrt{n}/2)) \leq c_0(\sqrt{n}/2)^p \rho_1^p. \quad (3.7)$$

We may cover E by a lattice of coordinate hypercubes of side ρ_1 , the number of which is at most

$$\left(\frac{\rho_2}{\rho_1} + 1\right) \dots \left(\frac{\rho_n}{\rho_1} + 1\right) \leq 2^n \frac{\rho_1 \dots \rho_n}{\rho_1^n}.$$

Summing the measure estimates of (3.7) over each of these hypercubes gives (3.6), with $c_1 = 2^n c_0 (\sqrt{n}/2)^p$. ■

Lemma 3.2 *Let s and p satisfy $0 \leq s < p \leq n$ with $p - s$ non-integral. Let ν_0 be a Borel measure with support in $B(0, \rho_0) \subset \mathbb{R}^n$ such that $\nu_0(B(x, r)) \leq c_0 r^p$ for all $x \in \mathbb{R}^n$ and $r > 0$, where c_0 is a constant. Then there exists $c < \infty$ depending on n, s, p, ρ_0 and c_0 such that, for all non-singular linear transformations $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int_{B(0, \rho_0)} \frac{d\nu_0(x)}{|Tx|^s} \leq c \frac{\alpha_1(T)^{n-p}}{\phi^{s+n-p}(T)}.$$

Proof. The result is clear if $s = 0$ so assume $s > 0$. For any Borel measure ν_0 and non-negative Borel measurable function f we have the decomposition

$$\int_X f(x) d\nu_0(x) = \int_0^\infty \nu_0\{x \in X : f(x) \geq u\} du,$$

see [11, Theorem 1.15]. Thus

$$\begin{aligned} \int_{B(0, \rho_0)} \frac{d\nu_0(x)}{|Tx|^s} &= \int_0^\infty \nu_0\{x : |x| \leq \rho_0 \text{ and } |Tx| \leq u^{-1/s}\} du \\ &= \int_0^\infty s t^{-s-1} \nu_0\{x : |x| \leq \rho_0 \text{ and } |Tx| \leq t\} dt \quad (\text{putting } t = u^{-1/s}) \end{aligned}$$

Let $\alpha_1, \dots, \alpha_n$ be the singular values of T and choose coordinate axes in the directions of mutually perpendicular eigenvectors of T^*T corresponding to the eigenvalues $\alpha_1^2, \dots, \alpha_n^2$. Then, writing $x = (x_1, \dots, x_n)$ with respect to these coordinates,

$$|Tx|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle = (\alpha_1^2 x_1^2 + \dots + \alpha_n^2 x_n^2),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^n .

Let m be the integer such that $s + n - p < m \leq s + n - p + 1$. Using Lemma 3.1 to bound the measure of the parallelepipeds, we get

$$\begin{aligned}
& \int_{B(0, \rho_0)} \frac{d\nu_0(x)}{|Tx|^s} \\
& \leq s \int_0^\infty t^{-s-1} \nu_0\{x : |x| \leq \rho_0 \text{ and } |\alpha_1^2 x_1^2 + \dots + \alpha_n^2 x_n^2| \leq t^2\} dt \\
& \leq s \int_0^{\alpha_m} t^{-s-1} \nu_0\left\{|x_1| \leq \frac{t}{\alpha_1}, \dots, |x_m| \leq \frac{t}{\alpha_m}, |x_{m+1}| \leq \rho_0, \dots, |x_n| \leq \rho_0\right\} dt \\
& \quad + s \int_{\alpha_m}^\infty t^{-s-1} \nu_0\left\{|x_1| \leq \frac{t}{\alpha_1}, \dots, |x_{m-1}| \leq \frac{t}{\alpha_{m-1}}, |x_m| \leq \rho_0, \dots, |x_n| \leq \rho_0\right\} dt \\
& \leq c_1 \int_0^{\alpha_m} \frac{\alpha_1^{n-p} \rho_0^{n-m} t^{m+p-n-s-1}}{\alpha_1 \dots \alpha_m} dt + c_2 \int_{\alpha_m}^\infty \frac{\alpha_1^{n-p} \rho_0^{n-m+1} t^{m+p-n-s-2}}{\alpha_1 \dots \alpha_{m-1}} dt, \\
& = c_3 \frac{t^{m+p-n-s} \alpha_1^{n-p}}{\alpha_1 \dots \alpha_m} \Big|_0^{\alpha_m} + c_4 \frac{t^{m+p-n-s-1} \alpha_1^{n-p}}{\alpha_1 \dots \alpha_{m-1}} \Big|_{\alpha_m}^\infty \\
& = c_3 \frac{\alpha_1^{n-p}}{\alpha_1 \dots \alpha_{m-1} \alpha_m^{s+n-p-m+1}} + c_4 \frac{\alpha_1^{n-p}}{\alpha_1 \dots \alpha_{m-1} \alpha_m^{s+n-p-m+1}} \\
& = c \frac{\alpha_1(T)^{n-p}}{\phi^{s+n-p}(T)}.
\end{aligned}$$

■

The following special case was proved directly in [4, Lemma 2.2].

Corollary 3.3 *Let s be non-integral with $0 < s < n$. Then there exists a constant $c < \infty$ depending on n , s and ρ_0 , such that, for all non-singular linear transformations $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int_{B(0, \rho_0)} \frac{dx}{|Tx|^s} dt \leq \frac{c}{\phi^s(T)}.$$

Proof. With ν_0 as n -dimensional Lebesgue measure, $\nu_0(B(x, r)) \leq c_0 r^n$, so the corollary is immediate from Lemma 3.2. ■

The following related estimate will be needed for the Fourier transform calculations.

Lemma 3.4 *Let s be non-integral with $0 < s < n$ and $s < \eta$. Then there exists a constant $c < \infty$ dependent on n , s and η , such that, for all non-singular linear transformations $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \frac{\min\{1, |Tt|^{-\eta}\}}{|t|^{n-s}} dt \leq \frac{c}{\phi^s(T)}.$$

Proof. Let T have singular values $\alpha_1, \dots, \alpha_n$. Choosing coordinate axes in the directions of mutually perpendicular eigenvectors of T^*T corresponding to the eigenvalues $\alpha_1^2, \dots, \alpha_n^2$, write $t = (t_1, \dots, t_n) = r(\theta_1, \dots, \theta_n) = r\theta$ in polar coordinates, where $r \geq 0$ and $\theta = (\theta_1, \dots, \theta_n)$ is a unit vector. Then

$$|Tt|^2 = \langle Tt, Tt \rangle = \langle t, T^*Tt \rangle = (\alpha_1^2 t_1^2 + \dots + \alpha_n^2 t_n^2) = r^2(\alpha_1^2 \theta_1^2 + \dots + \alpha_n^2 \theta_n^2).$$

Define $\omega(\theta)$ by $\omega(\theta)^2 = \alpha_1^2 \theta_1^2 + \dots + \alpha_n^2 \theta_n^2$ for unit vectors θ . Transforming into polar coordinates,

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\min\{1, |Tt|^{-\eta}\}}{|t|^{n-s}} dt &= \int_{\theta} \int_{r=0}^{\infty} \frac{\min\{1, r^{-\eta} \omega(\theta)^{-\eta}\}}{r^{n-s}} r^{n-1} dr d\theta \\
&= \int_{\theta} \left(\int_0^{\omega(\theta)^{-1}} r^{s-1} dr + \int_{\omega(\theta)^{-1}}^{\infty} r^{s-\eta-1} \omega(\theta)^{-\eta} dr \right) d\theta \\
&= c_1 \int_{\theta} \omega(\theta)^{-s} d\theta \quad (\text{since } s < \eta) \\
&= c_1(n-s) \int_{\theta} \int_{r=0}^1 r^{-s} \omega(\theta)^{-s} r^{n-1} dr d\theta \\
&= c_1(n-s) \int_{|t| \leq 1} \frac{dt}{|Tt|^s} \\
&\leq c\phi^s(T)^{-1},
\end{aligned}$$

by Corollary 3.3, where c_1 and c depend only on n, s and η . ■

4 Direct estimates of exceptional dimension

In this section we use a direct method to estimate the Hausdorff dimension of the exceptional sets $E(s) = \{\mathbf{a} : \dim_{\mathbb{H}} F(\mathbf{a}) < s\}$ where $0 < s \leq \min\{n, d(T_1, \dots, T_N)\}$.

We first consider the relationship between points \mathbf{a} in the parameter space \mathbb{R}^{nN} and the distance geometry of the attractors $F(\mathbf{a})$.

Let $\mathbf{i} \wedge \mathbf{j} = \mathbf{p} \in \mathbf{J}$, so that $\mathbf{i} = \mathbf{p}\mathbf{i}'$, and $\mathbf{j} = \mathbf{p}\mathbf{j}'$ with $\mathbf{i}', \mathbf{j}' \in \mathbf{J}_{\infty}$. From (2.5)

$$L_{\mathbf{i}, \mathbf{j}}(\mathbf{a}) = x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a}) = T_{\mathbf{i} \wedge \mathbf{j}}(x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a})) = T_{\mathbf{i} \wedge \mathbf{j}} L_{\mathbf{i}', \mathbf{j}'}(\mathbf{a}),$$

where $L \equiv L_{\mathbf{i}, \mathbf{j}} : \mathbb{R}^{nN} \rightarrow \mathbb{R}^n$ is the linear mapping given by

$$L(\mathbf{a}) = L_{\mathbf{i}, \mathbf{j}}(\mathbf{a}) = x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a}). \quad (4.8)$$

The quotient map $L' \equiv L'_{\mathbf{i}, \mathbf{j}} : \mathbb{R}^{nN} / \ker L \rightarrow \mathbb{R}^n$ given by $L'(\mathbf{a} + \ker L) = L(\mathbf{a})$ is a bijection provided that L is surjective, that is has rank n . The quotient norm on $\mathbb{R}^{nN} / \ker L$ is defined by $|\mathbf{a} + \ker L| = \inf\{|\mathbf{a} + \mathbf{k}| : \mathbf{k} \in \ker L\}$ and this induces a norm on L' in the usual way.

As with much of the work on self-affine sets, we require a condition on the maximum norm of the T_i

$$\gamma \equiv \max_{1 \leq i \leq N} \|T_i\| < \frac{1}{2}. \quad (4.9)$$

The following lemma is a variant of an estimate in [15, Proposition 3.1], first obtained in [4, Lemma 3.1] with $\frac{1}{3}$ replacing $\frac{1}{2}$.

Lemma 4.1 *Assume that the T_i satisfy (4.9). Then $L'_{\mathbf{i}, \mathbf{j}}$ is a bijection and there is a number $b > 0$ such that*

$$\|(L'_{\mathbf{i}, \mathbf{j}})^{-1}\| \leq b \quad (4.10)$$

for all $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$ such that $\mathbf{i} \wedge \mathbf{j} = \emptyset$.

Proof. Suppose without loss of generality that the first terms of the sequences \mathbf{i} and \mathbf{j} are 1 and 2 respectively. By (2.5)

$$\begin{aligned} L(\mathbf{a}) \equiv L_{\mathbf{i},\mathbf{j}}(\mathbf{a}) &= a_1 - a_2 + (T_1 a_{i_2} + T_1 T_{i_2} a_{i_3} + \cdots) \\ &\quad - (T_2 a_{j_2} + T_2 T_{j_2} a_{j_3} + \cdots) \\ &= a_1 - a_2 + L_1(a_1) + \cdots + L_N(a_N) \end{aligned} \quad (4.11)$$

where $\mathbf{a} = (a_1, a_2, \dots, a_N) \in \mathbb{R}^{nN}$ with $a_i \in \mathbb{R}^n$, where L_1, \dots, L_N are linear transformations on \mathbb{R}^n .

We may choose $l = 1$ or 2 such that for some $2 \leq m \leq \infty$ the indices i_k and j_k are not both equal to l for $1 \leq k < m$ and neither i_m nor j_m equals l if $m < \infty$. Suppose $l = 1$, the case $l = 2$ is identical except for a change in sign. Then

$$\|L_1\| \leq \sum_{k=2}^{m-1} \gamma^{k-1} + \sum_{k=m+1}^{\infty} 2\gamma^{k-1} \leq \gamma/(1-\gamma) < 1,$$

so by the usual result on inverses, $I + L_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible with $\|(I + L_1)^{-1}\| \leq (1-\gamma)/(1-2\gamma)$, with γ independent of \mathbf{i}, \mathbf{j} such that $\mathbf{i} \wedge \mathbf{j} = \emptyset$.

Given $y \in \mathbb{R}^n$, we may find $a_1 \in \mathbb{R}^n$ such that $(I + L_1)a_1 = y$, so $L(a_1, 0, \dots, 0) = y$ and $L'((a_1, 0, \dots, 0) + \ker L) = y$, giving that L' is surjective and so bijective. Moreover, for some constant c_1 depending only on the norms in use,

$$|(a_1, 0, \dots, 0) + \ker L| \leq c_1 |a_1| \leq c_1 |(I + L_1)^{-1}y| \leq b|y| \leq b|L'((a_1, 0, \dots, 0) + \ker L)|,$$

where $b = c_1(1-\gamma)/(1-2\gamma)$. Thus $\|(L')^{-1}\| \leq b$, as required. ■

A linear transformation $L : \mathbb{R}^{nN} \rightarrow \mathbb{R}^n$ induces a transformation on measures. Thus if ν is a Borel measure on \mathbb{R}^{nN} we get an image measure $L_{\#}\nu$ on \mathbb{R}^n defined by

$$(L_{\#}\nu)(A) = \nu(L^{-1}(A)) \quad (A \subset \mathbb{R}^n)$$

or equivalently by

$$\int f(x)d(L_{\#}\nu)(x) = \int f(L(\mathbf{a}))d\nu(\mathbf{a}) \quad (f \text{ continuous on } \mathbb{R}^n). \quad (4.12)$$

The following lemma estimates how L transforms the measures of balls.

Lemma 4.2 *Let ν be a Borel measure on \mathbb{R}^{nN} with support contained in some ball $B(0, \rho)$. Suppose that ν satisfies $\nu(B(\mathbf{a}, r)) \leq c_1 r^q$ for all $\mathbf{a} \in \mathbb{R}^{nN}$ and $r > 0$, where $n(N-1) \leq q \leq nN$. Let $R > 0$. Then for all L of full rank, $r > 0$ and $x \in \mathbb{R}^n$ with $0 < \|(L')^{-1}\|r \leq R$,*

$$(L_{\#}\nu)(B(x, r)) \leq c_2 \|(L')^{-1}\|^{q+n-nN} r^{q+n-nN} \quad (4.13)$$

where c_2 depends only on n, N, q, R and c_1 .

Proof. Let $x \in \mathbb{R}^n$ and let \mathbf{a} be the point of $(\ker L)^\perp$ such that $L(\mathbf{a}) = x$. Then

$$L^{-1}(B(x, r)) \subset ((\ker L)^\perp \cap B(\mathbf{a}, \|(L')^{-1}\|r)) \times \ker L \equiv A,$$

say, where the product is with respect to the orthogonal decomposition $\mathbb{R}^{nN} = (\ker L)^\perp \oplus \ker L$. Since ν is supported by $B(0, \rho)$, it is easy to see that A may be covered by at most $\lceil \rho / \|(L')^{-1}\|r \rceil^{nN-n}$ balls of radii $\sqrt{nN}\|(L')^{-1}\|r$, from which (4.13) follows. ■

The next lemma generalises [4, Lemma 3.1]. We write

$$\lambda = \max_{1 \leq i \leq N} \frac{\alpha_1(T_i)}{\alpha_n(T_i)}, \quad (4.14)$$

from which follows

$$\frac{\alpha_1(T_{\mathbf{i}})}{\alpha_n(T_{\mathbf{i}})} \leq \lambda^{|\mathbf{i}|} \quad (\mathbf{i} \in \mathbf{J}). \quad (4.15)$$

Lemma 4.3 *Let ν be a measure with support in $B(0, \rho) \subset \mathbb{R}^{nN}$ such that $\nu(B(\mathbf{a}, r)) \leq c_0 r^q$ for all $\mathbf{a} \in \mathbb{R}^{nN}$ and $r > 0$, where $n(N-1) < q \leq nN$. Let $0 < s < n$ be a number such that $nN - (n-s) < q \leq nN$ with $q-s$ non-integral. Assume that $\|T_i\| < \frac{1}{2}$ for all $1 \leq i \leq N$ and $\rho > 0$. Then there is a number c such that*

$$\int_{\mathbf{a} \in B(0, \rho)} \frac{d\nu(\mathbf{a})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} \leq c \lambda^{|\mathbf{i} \wedge \mathbf{j}|(nN-q)} \frac{1}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})}$$

for all distinct $\mathbf{i}, \mathbf{j} \in \mathbf{J}_\infty$.

Proof. For given $\mathbf{i} \neq \mathbf{j} \in \mathbf{J}_\infty$, writing $\mathbf{i} = \mathbf{p}\mathbf{i}'$ and $\mathbf{j} = \mathbf{p}\mathbf{j}'$ where $\mathbf{p} = \mathbf{i} \wedge \mathbf{j}$,

$$\begin{aligned} \int \frac{d\nu(\mathbf{a})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} &= \int_{\mathbf{a} \in B(0, \rho)} \frac{d\nu(\mathbf{a})}{|T_{\mathbf{i} \wedge \mathbf{j}}(x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a}))|^s} \\ &= \int_{\mathbf{a} \in B(0, \rho)} \frac{d\nu(\mathbf{a})}{|T_{\mathbf{i} \wedge \mathbf{j}} L_{\mathbf{i}', \mathbf{j}'}(\mathbf{a})|^s} \\ &= \int_{x \in B(0, \rho_0)} \frac{d\nu_0(x)}{|T_{\mathbf{i} \wedge \mathbf{j}}(x)|^s} \end{aligned}$$

using (4.12), where $\nu_0 = (L_{\mathbf{i}', \mathbf{j}'})_\# \nu$, with $\rho_0 = \rho \sup_{\mathbf{i}', \mathbf{j}'} \|L_{\mathbf{i}', \mathbf{j}'}\| < \infty$.

By Lemmas 4.1 and 4.2 $\nu_0(B(x, r)) = (L_{\mathbf{i}', \mathbf{j}'})_\# \nu(B(x, r)) \leq c_2 r^{q+n-nN}$ for all $\mathbf{i}', \mathbf{j}' \in \mathbf{J}$ with $\mathbf{i} \wedge \mathbf{j}$, so taking $p = q + n - nN$ in Lemma 3.2,

$$\int \frac{d\nu(\mathbf{a})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} \leq c \frac{\alpha_1(T_{\mathbf{i} \wedge \mathbf{j}})^{nN-q}}{\phi^{s+nN-q}(T_{\mathbf{i} \wedge \mathbf{j}})} \quad (4.16)$$

$$\begin{aligned} &\leq \left(\frac{\alpha_1(T_{\mathbf{i} \wedge \mathbf{j}})}{\alpha_n(T_{\mathbf{i} \wedge \mathbf{j}})} \right)^{nN-q} \frac{c}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})} \\ &\leq c \lambda^{|\mathbf{i} \wedge \mathbf{j}|(nN-q)} \phi^s(T_{\mathbf{i} \wedge \mathbf{j}})^{-1}, \end{aligned} \quad (4.17)$$

where c is independent of \mathbf{i} and \mathbf{j} . ■

The next lemma indicates how such integrals are used to get almost sure estimates for $\dim_{\mathbb{H}} F(\mathbf{a})$.

Lemma 4.4 *Let ν be a measure with support in $B(0, \rho) \subset \mathbb{R}^{nN}$ such that $0 < \nu(B(0, \rho)) < \infty$. Let μ be a Borel measure on \mathbf{J}_∞ with $0 < \mu(\mathbf{J}_\infty) < \infty$ such that for some $0 \leq s < n$*

$$\int_{\mathbf{J}_\infty} \int_{\mathbf{J}_\infty} \int_{\mathbf{a} \in B(0, \rho)} \frac{d\nu(\mathbf{a}) d\mu(\mathbf{i}) d\mu(\mathbf{j})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} < \infty. \quad (4.18)$$

Then $\dim_{\mathbb{H}} F(\mathbf{a}) \geq s$ for ν -almost all $\mathbf{a} \in B(0, \rho)$.

Proof. Applying Fubini's Theorem to (4.18) we conclude that for ν -almost all $\mathbf{a} \in B(0, \rho)$

$$\int_{\mathbf{J}_\infty} \int_{\mathbf{J}_\infty} \frac{d\mu(\mathbf{i})d\mu(\mathbf{j})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} < \infty.$$

For each \mathbf{a} , we may define a measure $\mu_{\mathbf{a}}$ on \mathbb{R}^n by

$$\mu_{\mathbf{a}}(E) = \mu\{\mathbf{i} : x_{\mathbf{i}}(\mathbf{a}) \in E\}$$

or equivalently by

$$\int f(x)d\mu_{\mathbf{a}}(x) = \int f(x_{\mathbf{i}}(\mathbf{a}))d\mu(\mathbf{i}), \quad (4.19)$$

for all continuous bounded f on \mathbb{R}^n . Since the mapping $\mathbf{J}_\infty \rightarrow \mathbb{R}^n$ given by $\mathbf{i} \mapsto x_{\mathbf{i}}(\mathbf{a})$ is continuous, it follows that $\mu_{\mathbf{a}}$ is a Borel measure on supported by $F(\mathbf{a})$, the image of \mathbf{J}_∞ under this mapping, with $0 < \mu_{\mathbf{a}}(\mathbb{R}^n) = \mu(\mathbf{J}_\infty) < \infty$. Thus for ν -almost all \mathbf{a} the attractor $F(\mathbf{a})$ supports a mass distribution of finite s -energy, so $\dim_{\text{H}} F(\mathbf{a}) \geq s$, by the standard energy criterion [3, Corollary 6.6] or [11, Chapter 8]. ■

In order to estimate these energy integrals, we set up measures of Hausdorff type on \mathbf{J}_∞ . The following proposition summarises the construction of the measures and gives a version of Frostman's lemma that yields measures with bounded densities.

Proposition 4.5 *Let $\psi : \mathbf{J} \rightarrow \mathbb{R}$ satisfy $\psi(\mathbf{i}) > 0$ for all $\emptyset \neq \mathbf{i} \in \mathbf{J}$ and $\psi(\emptyset) = 0$, and $\lim_{k \rightarrow \infty} \max\{\psi(\mathbf{i}|_k) : |\mathbf{i}| = k\} \rightarrow 0$. Then we may define a regular Borel measure \mathcal{M} using Carathéodory's construction by setting*

$$\mathcal{M}_k(A) = \inf \left\{ \sum_{\mathbf{i}} \psi(\mathbf{i}) : A \subset \bigcup \mathcal{C}_{\mathbf{i}}, |\mathbf{i}| \geq k \right\}$$

and

$$\mathcal{M}(A) = \lim_{k \rightarrow \infty} \mathcal{M}_k(A),$$

for all $A \subset \mathbf{J}_\infty$. If $\mathcal{M}(\mathbf{J}_\infty) > 0$ then there exists a Borel measure μ on \mathbf{J}_∞ such that $0 < \mu(\mathbf{J}_\infty) < \infty$ and, for some $c > 0$,

$$\mu(\mathcal{C}_{\mathbf{i}}) \leq c\psi(\mathbf{i}) \quad (\mathbf{i} \in \mathbf{J}). \quad (4.20)$$

Proof. Carathéodory's construction and its properties are described in detail in [11, 14].

If $\mathcal{M}(\mathbf{J}_\infty) > 0$ then, by what is now a routine argument, there exists a compact $E \subset \mathbf{J}_\infty$ such that $\mathcal{M}(E) > 0$ and $\mathcal{M}(E \cap \mathcal{C}_{\mathbf{i}}) \leq c\psi(\mathbf{i})$ for all $\mathbf{i} \in \mathbf{J}$. (This is essentially a special case of [14, Theorem 54] or may be proved similarly to [3, Theorem 5.4] or [11, Theorem 8.8].) Thus the measure μ defined by $\mu(A) = \mathcal{M}(E \cap A)$ for $A \subset \mathbf{J}_\infty$ has the desired properties. ■

We specialise to measures of particular interest in our context. Recall from (4.14) that $\lambda = \max_{1 \leq i \leq N} \alpha_1(T_i)/\alpha_n(T_i) \geq 1$. Fix $q \in \mathbb{R}$ and $s > 0$. By Proposition 4.5 we get Borel measures $\mu_k^{q,s}$ on \mathbf{J}_∞ by setting, for each positive integer k ,

$$\mu_k^{q,s}(A) = \inf \left\{ \sum_{\mathbf{i}} \lambda^{|\mathbf{i}|(q-nN)} \phi^s(T_{\mathbf{i}}) : A \subset \bigcup_{\mathbf{i}} \mathcal{C}_{\mathbf{i}}, |\mathbf{i}| \geq k \right\} \quad (A \subset \mathbf{J}_\infty), \quad (4.21)$$

and then

$$\mu^{q,s}(A) = \lim_{k \rightarrow \infty} \mu_k^{q,s}(A) \quad (A \subset \mathbf{J}_\infty). \quad (4.22)$$

In particular, if $\mu^{q,s}(\mathbf{J}_\infty) > 0$ then there is a measure μ with $0 < \mu(\mathbf{J}_\infty) < \infty$ and $c > 0$ such that

$$\mu(\mathbf{C}_i) \leq c \lambda^{|\mathbf{i}|(q-nN)} \phi^s(T_i). \quad (4.23)$$

Recall that $\psi : \mathbf{J}_\infty \rightarrow \mathbb{R}$ is *submultiplicative* if

$$\psi(\mathbf{ij}) \leq \psi(\mathbf{i})\psi(\mathbf{j}) \quad (\mathbf{i}, \mathbf{j} \in \mathbf{J}_\infty). \quad (4.24)$$

Since $\phi(T_i)$ is submultiplicative, so is $\psi(\mathbf{i}) = \lambda^{|\mathbf{i}|(q-nN)} \phi^s(T_i)$. The next two propositions which relate measures to critical parameters for convergence of series are consequences of this submultiplicativity.

Proposition 4.6 *There exists a unique number $d \equiv d(T_1, \dots, T_N) > 0$ such that*

$$\lim_{k \rightarrow \infty} \left[\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^d(T_i) \right]^{1/k} = 1.$$

Moreover,

- (a) $\sum_{\mathbf{i} \in \mathbf{J}} \phi^s(T_i) < \infty$ if $s > d$, and $\sum_{\mathbf{i} \in \mathbf{J}} \phi^s(T_i) = \infty$ if $0 \leq s < d$,
- (b) $\mu^{nN,s}(\mathbf{J}_\infty) = 0$ if $s > d$, and $\mu^{nN,s}(\mathbf{J}_\infty) = \infty$ if $0 \leq s < d$.

Proof. This is a restatement of [4, Proposition 4.1] and depends on $\phi^s(T_i)$ being submultiplicative, along with the fact that

$$\phi^s(T_i) \alpha_+^{l|\mathbf{i}|} \geq \phi^{s+l}(T_i) \geq \phi^s(T_i) \alpha_-^{l|\mathbf{i}|} \quad (\mathbf{i} \in \mathbf{J}) \quad (4.25)$$

for $l \geq 0$ which gives equicontinuity and strict monotonicity of $[\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_i)]^{1/k}$. ■

If we now assume that the T_i are not all similarities, that is if $\lambda > 1$, we may define numbers

$$q_s = nN - \frac{\log |\lim_{k \rightarrow \infty} [\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_i)]^{1/k}|}{\log \lambda} \quad (s > 0). \quad (4.26)$$

Then $q_d = Nn$ and q_s is strictly increasing and continuous for $0 \leq s \leq n$.

We get the following analogue of Proposition 4.6.

Proposition 4.7 *Let $0 \leq s \leq n$. Then:*

- (a) $\lim_{k \rightarrow \infty} [\lambda^{k(q_s - nN)} \sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_i)]^{1/k} = 1$,
- (b) $\sum_{\mathbf{i} \in \mathbf{J}} \lambda^{|\mathbf{i}|(q-nN)} \phi^s(T_i) < \infty$ if $q < q_s$, and $\sum_{\mathbf{i} \in \mathbf{J}} \lambda^{|\mathbf{i}|(q-nN)} \phi^s(T_i) = \infty$ if $q > q_s$.
- (c) $\mu^{q,s}(\mathbf{J}_\infty) = 0$ if $q < q_s$, and $\mu^{q,s}(\mathbf{J}_\infty) = \infty$ if $q > q_s$.

Proof. Part (a) follows from the definition of q_s . The remainder is very similar to [4, Proposition 4.1] and depends on the fact that $\psi(\mathbf{i}) \equiv \lambda^{|\mathbf{i}|(q-nN)} \phi^s(T_i)$ is submultiplicative together with (4.25). ■

The following corollary gives a tractable upper bound for q_s .

Corollary 4.8 *If T_i ($i = 1, \dots, N$) are not all similarities, then*

$$q_s \leq nN - (d - s) \frac{|\log \alpha_+|}{\log \lambda} \quad (0 < s \leq d). \quad (4.27)$$

Proof. Using (4.25) we get for $s \leq d$

$$\left(\sum_{\mathbf{J}_k} \phi^d(T_{\mathbf{i}})\right)^{1/k} \leq \left(\sum_{\mathbf{J}_k} \phi^s(T_{\mathbf{i}})\right)^{1/k} \alpha_+^{(d-s)}$$

so using that $\lim_{k \rightarrow \infty} \left(\sum_{\mathbf{J}_k} \phi^d(T_{\mathbf{i}})\right)^{1/k} = 1$, and taking logarithms,

$$0 \leq \log \left(\lim_{k \rightarrow \infty} \left(\sum_{\mathbf{J}_k} \phi^d(T_{\mathbf{i}})\right)^{1/k} \right) + (d-s) \log \alpha_+$$

which gives (4.27). ■

We put together these properties to obtain estimates for the Hausdorff dimension of the exceptional sets

$$E(s) = \{\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^{nN} : \dim_{\mathbb{H}} F(\mathbf{a}) < s\}. \quad (4.28)$$

Theorem 4.9 *Assume that $\|T_i\| < \frac{1}{2}$ for all i . Let $0 < s \leq \max\{n, d(T_1, \dots, T_N)\}$, and let q_s be given by (4.26). Then*

$$\dim_{\mathbb{H}} E(s) \leq \max\{nN - (n-s), q_s\}, \quad (4.29)$$

so in particular

$$\dim_{\mathbb{H}} E(s) \leq \max \left\{ nN - (n-s), nN - (d-s) \frac{|\log \alpha_+|}{\log \lambda} \right\}. \quad (4.30)$$

Proof. Since $q_d = nN$, we may assume that $0 < s < d \equiv d(T_1, \dots, T_N) \leq n$, otherwise the result is trivial. For a contradiction, suppose that for some $0 < s \leq d \leq n$ the conclusion is false. Then there exists a number q with $\max\{nN - (n-s), q_s\} < q < nN$ and $q - s$ non-integral such that $\mathcal{H}^q(E(s)) > 0$. By Frostman's lemma [11, Theorem 8.8] there exists a Borel measure ν supported by $E(s)$ with $\nu(E(s)) > 0$ such that $\nu(B(\mathbf{a}, r)) \leq c_0 r^q$ for all $\mathbf{a} \in \mathbb{R}^{nN}$ and $r > 0$, and we may further assume that ν has bounded support in $B(0, \rho)$, say.

Since $q > q_s$, we may find t with $s < t < d$ such that $q_s < q_t < q$, so $\mu^{q,t}(\mathbf{J}_{\infty}) = \infty$ by Proposition 4.7, so by (4.23) there is a Borel measure μ on \mathbf{J}_{∞} with $\mu(\mathbf{J}_{\infty}) > 0$ such that

$$\mu(\mathcal{C}_{\mathbf{i}}) \leq c_1 \lambda^{|\mathbf{i}|(q-nN)} \phi^t(T_{\mathbf{i}}) \quad (\mathbf{i} \in \mathbf{J}). \quad (4.31)$$

Since $q > nN - (n - s)$ Lemma 4.3 gives

$$\begin{aligned}
\int_{\mathbf{J}_\infty} \int_{\mathbf{J}_\infty} \int_{\mathbf{a} \in B(0, \rho)} \frac{d\nu(\mathbf{a}) d\mu(\mathbf{i}) d\mu(\mathbf{j})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} &\leq c \int_{\mathbf{J}_\infty} \int_{\mathbf{J}_\infty} \lambda^{|\mathbf{i} \wedge \mathbf{j}|(nN-q)} \frac{d\mu(\mathbf{i}) d\mu(\mathbf{j})}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})} \quad (4.32) \\
&\leq c \sum_{\mathbf{p} \in \mathbf{J}} \sum_{\mathbf{i}' \wedge \mathbf{j}' = \emptyset} \lambda^{|\mathbf{p}|(nN-q)} \frac{\mu(\mathcal{C}_{\mathbf{p}\mathbf{i}'}) \mu(\mathcal{C}_{\mathbf{p}\mathbf{j}'})}{\phi^s(T_{\mathbf{p}})} \\
&\leq c \sum_{\mathbf{p} \in \mathbf{J}} \lambda^{|\mathbf{p}|(nN-q)} \frac{\mu(\mathcal{C}_{\mathbf{p}})^2}{\phi^s(T_{\mathbf{p}})} \\
&\leq cc_1 \sum_{k=0}^{\infty} \sum_{\mathbf{p} \in \mathbf{J}_k} \frac{\phi^t(T_{\mathbf{p}}) \mu(\mathcal{C}_{\mathbf{p}})}{\phi^s(T_{\mathbf{p}})} \\
&\leq cc_1 \sum_{k=0}^{\infty} \sum_{\mathbf{p} \in \mathbf{J}_k} \alpha_+^{k(t-s)} \mu(\mathcal{C}_{\mathbf{p}}) \\
&\leq cc_1 \mu(E) \sum_{k=0}^{\infty} \alpha_+^{k(t-s)} \\
&< \infty,
\end{aligned}$$

where $\alpha_+ = \max_{1 \leq i \leq N} \{\alpha_1(T_i)\} < 1$ and the series $\sum_{k=0}^{\infty} \alpha_+^{k(t-s)}$ is convergent since $t > s$. By Lemma 4.4, $\dim_{\mathbb{H}} F(\mathbf{a}) \geq s$ for ν -almost all $\mathbf{a} \in A$, which contradicts the definition of $E(s)$, since $\nu(E(s)) > 0$. The conclusion follows. ■

Note that it would have been possible to get what in principle is a better bound in (4.29) by using the estimate (4.16) rather than (4.17) at (4.32). Then the premeasure $\mu_k^{q,t}$ at (4.21) would be replaced by

$$\mu_k^{q,s}(A) = \inf \left\{ \sum_{\mathbf{i}} \alpha_1(T_{\mathbf{i}})^{nN-q} / \phi^{s+nN-q}(T_{\mathbf{i}}) : A \subset \bigcup_{\mathbf{i}} \mathcal{C}_{\mathbf{i}}, |\mathbf{i}| \geq k \right\} \quad (A \subset \mathbf{J}_\infty)$$

with q_s defined as the infimum q such that $\mu_k^{q,s}(\mathbf{J}_\infty) < \infty$. However, since $\psi(\mathbf{i}) = \alpha_1(T_{\mathbf{i}})^{nN-q} / \phi^{s+nN-q}(T_{\mathbf{i}})$ is not in general submultiplicative, there is no analogue of Proposition 4.7 and q_s need not be characterised as the critical parameter for convergence of a series.

We may also get an estimate for the exceptional dimension when the T_i are all similarities. In this case, $d = d(T_1, \dots, T_N)$ is just the similarity dimension of the IFS given by $\sum_{i=1}^N \alpha(T_i)^d = 1$, where $\alpha(T)$ is the scale ratio of the similarity T , which equals all the singular values of T .

Corollary 4.10 *Suppose that the transformations T_i are all similarities with $\|T_i\| < \frac{1}{2}$ for all i and similarity dimension $0 < d < n$. Then*

$$\dim_{\mathbb{H}} \{ \mathbf{a} = (a_1, \dots, a_N) : \dim_{\mathbb{H}} F(\mathbf{a}) < d \} \leq nN - n + d.$$

Proof. The argument of Theorem 4.9 holds for any $\lambda \geq \max_{1 \leq i \leq N} \{\alpha_1(T_i) / \alpha_n(T_i)\}$. Thus we can take λ arbitrarily close to 1 in (4.30) to get the result.

Alternatively, the proof of Theorem 4.9 goes through by working with $\lambda = 1$ and using the measure $\mu^{nN,t}$ from Proposition 4.6 for $s < t < d$. ■

5 Fourier transform estimates of exceptional dimensions

We introduce the Fourier transform properties needed to study the Fourier dimension of the exceptional sets. The n -dimensional Fourier transform \widehat{f} of a Lebesgue integrable function f on \mathbb{R}^n is defined by

$$\widehat{f}(t) = \int e^{-i\langle t, x \rangle} f(x) dx \quad (t \in \mathbb{R}^n),$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbb{R}^n . The Fourier transform is extended in the usual way to larger classes of function. Similarly, the Fourier transform $\widehat{\mu}$ of a Borel measure μ on \mathbb{R}^n is given by

$$\widehat{\mu}(t) = \int e^{-i\langle t, x \rangle} d\mu(x) \quad (t \in \mathbb{R}^n).$$

Let μ be a mass distribution (i.e. a positive finite Borel measure) on \mathbb{R}^n . The s -energy of μ is

$$I_s(\mu) = \int \int \frac{d\mu(x)d\mu(y)}{|x-y|^s}. \quad (5.33)$$

It is well-known that if a set $F \subset \mathbb{R}^n$ supports a mass distribution μ such that $I_s(\mu) < \infty$ then its Hausdorff dimension $\dim_{\mathbb{H}} F \geq s$. We may transform (5.33) to get

$$I_s(\mu) = c_0 \int \frac{|\widehat{\mu}(t)|^2}{|t|^{n-s}} dt, \quad (5.34)$$

for $0 < s < n$, where c_0 depends only on n and s . Formally (5.34) follows from Parseval's theorem and that the Fourier transform of $|x|^{-s}$ is $c_1|t|^{-(n-s)}$, see [11] for a rigorous derivation.

The Fourier dimension $\dim_{\mathbb{F}} A$ of $A \subset \mathbb{R}^n$ is given by

$$\dim_{\mathbb{F}} A = \sup\{0 \leq s \leq n : \text{there exists a mass distribution } \mu \text{ on } A \\ \text{such that } \widehat{\mu}(t) = O(|t|^{-s/2}) \text{ as } t \rightarrow \infty\}.$$

For every Borel set $A \subset \mathbb{R}^n$, $\dim_{\mathbb{F}} A \leq \dim_{\mathbb{H}} A$, with equality for sets known as Salem sets a class that includes many sets especially those involving a random construction, see [1, 10, 11].

As before, let $F(\mathbf{a})$ be the self-affine set

$$F(\mathbf{a}) = \bigcup_{i=1}^N S_i(F(\mathbf{a})) \quad \text{where} \quad S_i(x) = T_i(x) + a_i \quad i = 1, \dots, N$$

where $\mathbf{a} = (a_1, \dots, a_N)$. Let μ be a measure on the code space \mathbf{J}_{∞} , and $\mu_{\mathbf{a}}$ be the induced measure on $F(\mathbf{a})$ given by (4.19). Then

$$\widehat{\mu}_{\mathbf{a}}(t) = \int \exp\{-i\langle t, x_{\mathbf{i}}(\mathbf{a}) \rangle\} d\mu(\mathbf{i}),$$

and

$$\overline{\widehat{\mu}_{\mathbf{a}}(t)} = \int \exp\{i\langle t, x_{\mathbf{i}}(\mathbf{a}) \rangle\} d\mu(\mathbf{i}).$$

Given $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$, if $\mathbf{i} \wedge \mathbf{j} = \mathbf{p} \in \mathbf{J}$, let $\mathbf{i} = \mathbf{p}\mathbf{i}'$, and $\mathbf{j} = \mathbf{p}\mathbf{j}'$ where $\mathbf{i}', \mathbf{j}' \in \mathbf{J}_{\infty}$. Then

$$|\widehat{\mu}_{\mathbf{a}}(t)|^2 = |\widehat{\mu}_{\mathbf{a}}(t)| \overline{|\widehat{\mu}_{\mathbf{a}}(t)|} = \int \int \exp\{i\langle t, T_{\mathbf{i} \wedge \mathbf{j}}(x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a})) \rangle\} d\mu(\mathbf{i}) d\mu(\mathbf{j}),$$

where $x_{\mathbf{i}}(\mathbf{a})$ is as in (2.5). Thus we get an expression for the s -energy $I^s(\mu_{\mathbf{a}})$ for $0 < s < n$ from (5.34):

$$I_s(\mu_{\mathbf{a}}) = c_0 \int_t \int_{\mathbf{i}} \int_{\mathbf{j}} \frac{\exp i\langle t, T_{\mathbf{i} \wedge \mathbf{j}}(x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a})) \rangle}{|t|^{n-s}} d\mu(\mathbf{i}) d\mu(\mathbf{j}) dt. \quad (5.35)$$

For each $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$ and $R > 0$, write

$$Q_{\mathbf{i}, \mathbf{j}}^R(\mathbf{a}) = \int_{|t| \leq R} \frac{\exp i\langle t, T_{\mathbf{i} \wedge \mathbf{j}}(x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a})) \rangle}{|t|^{n-s}} dt.$$

We estimate (5.35) for certain sets of $\mathbf{a} \in \mathbb{R}^{nN}$ by first considering $Q_{\mathbf{i}, \mathbf{j}}^R(\mathbf{a})$.

Lemma 5.1 *Let s be non-integral such that $0 < s < n$ and let $\eta > s$. Let ν be a mass distribution on \mathbb{R}^{nN} satisfying*

$$|\widehat{\nu}(\mathbf{s})| \leq a|\mathbf{s}|^{-\eta} \quad (\mathbf{s} \in \mathbb{R}^{nN}) \quad (5.36)$$

(where the Fourier transform of ν is defined on the parameter space \mathbb{R}^{nN}). Assume that $\|T_i\| < \frac{1}{2}$ for all i . Then there is a constant b such that

$$\left| \int_{\mathbf{a} \in \mathbb{R}^{nN}} Q_{\mathbf{i}, \mathbf{j}}^R(\mathbf{a}) d\nu(\mathbf{a}) \right| \leq \frac{b}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})}$$

for all $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$ and $R > 0$.

Proof. For each \mathbf{i}, \mathbf{j} we consider the transformation $L \equiv L_{\mathbf{i}', \mathbf{j}'} : \mathbb{R}^{nN} \rightarrow \mathbb{R}^n$ given by

$$L(\mathbf{a}) = L_{\mathbf{i}', \mathbf{j}'}(\mathbf{a}) = x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a}),$$

where as usual $\mathbf{i} = \mathbf{p}\mathbf{i}'$ and $\mathbf{j} = \mathbf{p}\mathbf{j}'$ where $\mathbf{i} \wedge \mathbf{j} = \mathbf{p}$. As in Lemma 4.1, provided that $\|T_i\| < \frac{1}{2}$ for all $i = 1, \dots, n$, we have that L is of full rank n with $\|L^{-1}\| \geq c_1$ for some c_1 independent of \mathbf{i}, \mathbf{j} , where we regard $L^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^{nN} / \ker L$ in the natural way. We write $L^* : \mathbb{R}^n \rightarrow \mathbb{R}^{nN}$ for the adjoint or dual mapping to L defined by

$$\langle L(\mathbf{a}), x \rangle = \langle \mathbf{a}, L^*(x) \rangle_{nN} \quad (\mathbf{a} \in \mathbb{R}^{nN}, x \in \mathbb{R}^n)$$

where $\langle \cdot, \cdot \rangle_{nN}$ denotes the inner product on \mathbb{R}^{nN} . (Thus with respect to any pair of bases, the matrix of L^* is the transpose of that of L .) Note also that the image of L^* is the orthogonal complement $(\ker L)^{\perp}$, and that if we regard $(L^*)^{-1} : (\ker L)^{\perp} \rightarrow \mathbb{R}^n$ then $\|(L^*)^{-1}\| = \|L^{-1}\| \geq c_1$.

Thus, writing $T_{\mathbf{i}\wedge\mathbf{j}}^* : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for the transpose of $T_{\mathbf{i}\wedge\mathbf{j}}$,

$$\begin{aligned}
\left| \int_{\mathbf{a}} Q_{\mathbf{i}\mathbf{j}}^R(\mathbf{a}) d\nu(\mathbf{a}) \right| &= \left| \int_{\mathbf{a}} \int_{|t|\leq R} \frac{\exp i\langle t, T_{\mathbf{i}\wedge\mathbf{j}}(L(\mathbf{a})) \rangle}{|t|^{n-s}} dt d\nu(\mathbf{a}) \right| \\
&= \left| \int_{\mathbf{a}} \int_{|t|\leq R} \frac{\exp i\langle T_{\mathbf{i}\wedge\mathbf{j}}^* t, L(\mathbf{a}) \rangle}{|t|^{n-s}} dt d\nu(\mathbf{a}) \right| \\
&= \left| \int_{|t|\leq R} \int_{\mathbf{a}} \frac{\exp i\langle L^* T_{\mathbf{i}\wedge\mathbf{j}}^* t, \mathbf{a} \rangle_{nN}}{|t|^{n-s}} d\nu(\mathbf{a}) dt \right| \\
&= \left| \int_{|t|\leq R} \frac{\widehat{\nu}(-L^* T_{\mathbf{i}\wedge\mathbf{j}}^* t)}{|t|^{n-s}} dt \right| \\
&\leq \int_{|t|\leq R} \frac{\min\{\nu(\mathbb{R}^{nN}), a|L^* T_{\mathbf{i}\wedge\mathbf{j}}^* t|^{-\eta}\}}{|t|^{n-s}} dt \\
&\leq c_2 \int_{|t|\leq R} \frac{\min\{1, |T_{\mathbf{i}\wedge\mathbf{j}}^* t|^{-\eta}\}}{|t|^{n-s}} dt \\
&\leq \frac{b}{\phi^s(T_{\mathbf{i}\wedge\mathbf{j}}^*)},
\end{aligned}$$

where c_2 and b are independent of \mathbf{i}, \mathbf{j} and R , and we have used Fubini's theorem, justified since the integrand is absolutely integrable over $|t| \leq R$, along with (5.36) and Lemma 3.4. Since $\phi^s(T_{\mathbf{i}\wedge\mathbf{j}}^*) = \phi^s(T_{\mathbf{i}\wedge\mathbf{j}})$ the conclusion follows. ■

Lemma 5.2 *Let s be non-integral such that $0 < s < n$ and let $\eta > s$. Suppose that $\|T_i\| < \frac{1}{2}$ for all i and let ν be a mass distribution on \mathbb{R}^{nN} satisfying*

$$|\widehat{\nu}(\mathbf{s})| \leq a|\mathbf{s}|^{-\eta} \quad (\mathbf{s} \in \mathbb{R}^{nN}). \quad (5.37)$$

Then for all s there is a constant $b < \infty$ such that

$$\int_{\mathbf{a} \in \mathbb{R}^{nN}} I_s(\mu_{\mathbf{a}}) d\nu(\mathbf{a}) \leq b \int_{\mathbf{j}} \int_{\mathbf{i}} \phi^s(T_{\mathbf{i}\wedge\mathbf{j}})^{-1} d\mu(\mathbf{i}) d\mu(\mathbf{j}).$$

Proof. Fubini's Theorem gives

$$\begin{aligned}
\int_{|t|\leq R} \int_{\mathbf{a}} \frac{|\widehat{\mu}_{\mathbf{a}}(t)|^2}{|t|^{n-s}} d\nu(\mathbf{a}) dt &= \int_{|t|\leq R} \int_{\mathbf{a}} \int_{\mathbf{j}} \int_{\mathbf{i}} \frac{\exp i\langle t, T_{\mathbf{i}\wedge\mathbf{j}}(x_{\mathbf{Y}}(\mathbf{a}) - x_{\mathbf{Y}'}(\mathbf{a})) \rangle}{|t|^{n-s}} d\mu(\mathbf{i}) d\mu(\mathbf{j}) d\nu(\mathbf{a}) dt \\
&= \int_{\mathbf{j}} \int_{\mathbf{i}} \int_{\mathbf{a}} Q_{\mathbf{i}\mathbf{j}}^R(\mathbf{a}) d\nu(\mathbf{a}) d\mu(\mathbf{i}) d\mu(\mathbf{j}), \quad (5.38)
\end{aligned}$$

noting that the integrand is dominated by $|t|^{-(n-s)}$ which is integrable with respect to the product measure over $|t| \leq R$. Hence, by Lemma 5.1

$$\begin{aligned}
\int_{|t|\leq R} \int_{\mathbf{a}} \frac{|\widehat{\mu}_{\mathbf{a}}(t)|^2}{|t|^{n-s}} d\nu(\mathbf{a}) dt &\leq \int_{\mathbf{j}} \int_{\mathbf{i}} \left| \int_{\mathbf{a}} Q_{\mathbf{i}\mathbf{j}}^R(\mathbf{a}) d\nu(\mathbf{a}) \right| d\mu(\mathbf{i}) d\mu(\mathbf{j}) \\
&\leq b \int_{\mathbf{j}} \int_{\mathbf{i}} \frac{d\mu(\mathbf{i}) d\mu(\mathbf{j})}{\phi^s(T_{\mathbf{i}\wedge\mathbf{j}})},
\end{aligned}$$

where b is independent of $R > 0$. Applying Fubini's theorem again followed by the monotone convergence theorem,

$$\begin{aligned} \int_{\mathbf{a} \in \mathbb{R}^{nN}} \int_{t \in \mathbb{R}^n} \frac{|\widehat{\mu}_{\mathbf{a}}(t)|^2}{|t|^{n-s}} dt d\nu(\mathbf{a}) &= \int_{t \in \mathbb{R}^n} \int_{\mathbf{a} \in \mathbb{R}^{nN}} \frac{|\widehat{\mu}_{\mathbf{a}}(t)|^2}{|t|^{n-s}} d\nu(\mathbf{a}) dt \\ &= \lim_{R \rightarrow \infty} \int_{|t| \leq R} \int_{\mathbf{a}} \frac{|\widehat{\mu}_{\mathbf{a}}(t)|^2}{|t|^{n-s}} d\nu(\mathbf{a}) dt \\ &\leq b \int_{\mathbf{j}} \int_{\mathbf{i}} \frac{1}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})} d\mu(\mathbf{i}) d\mu(\mathbf{j}), \end{aligned}$$

and the conclusion follows from (5.34). ■

Theorem 5.3 *Assume that $\|T_i\| < \frac{1}{2}$ for all i . For $0 < s \leq \min\{n, d(T_1, \dots, T_N)\}$ the exceptional set*

$$E(s) = \{\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^{nN} : \dim_{\mathbb{H}} F(\mathbf{a}) < s\}$$

has Fourier dimension satisfying

$$\dim_{\mathbb{F}} E(s) \leq 2s.$$

(Note that $2s \leq Nn$.)

Proof. It is enough to prove this for s non-integral with $0 < s < \min\{n, d(T_1, \dots, T_N)\}$ and extend to other cases by approximation. Suppose, for a contradiction, that $2s < \dim_{\mathbb{F}} E(s) \leq nN$. Then we can find a measure ν supported by $E(s) \subset \mathbb{R}^{nN}$ and $a > 0$ such that $0 < \nu(E(s)) < \infty$ and $|\widehat{\nu}(\mathbf{s})| \leq a|\mathbf{s}|^{-\eta}$ for $\mathbf{s} \in \mathbb{R}^{nN}$, for some $s < \eta < \frac{1}{2} \dim_{\mathbb{F}} E(s)$.

Choose t such that $s < t < \min\{n, d(T_1, \dots, T_N)\}$. By Proposition 4.6 $\mu^{nN,t}(\mathbf{J}) = \infty$ so by (4.23) there is a measure μ on \mathbf{J}_{∞} such that

$$\mu(\mathcal{C}_{\mathbf{i}}) \leq c_1 \phi^t(T_{\mathbf{i}}) \quad (\mathbf{i} \in \mathbf{J}).$$

For $\mathbf{i}, \mathbf{j} \in \mathbf{J}_{\infty}$ write, as before, $\mathbf{i} \wedge \mathbf{j} = \mathbf{p}$ and $\mathbf{i} = \mathbf{p}\mathbf{i}'$ and $\mathbf{j} = \mathbf{p}\mathbf{j}'$. Since $s < \eta$ and s is non-integral, Lemma 5.2 gives

$$\begin{aligned} \int_{\mathbf{a} \in \mathbb{R}^{nN}} I_s(\mu_{\mathbf{a}}) d\nu(\mathbf{a}) &\leq b \int_{\mathbf{i}} \int_{\mathbf{j}} \phi^s(T_{\mathbf{i} \wedge \mathbf{j}})^{-1} d\mu(\mathbf{i}) d\mu(\mathbf{j}) \\ &\leq b \sum_{\mathbf{p} \in \mathbf{J}} \sum_{\mathbf{i}' \wedge \mathbf{j}' = \emptyset} \frac{\mu(\mathcal{C}_{\mathbf{p}\mathbf{i}'}) \mu(\mathcal{C}_{\mathbf{p}\mathbf{j}'})}{\phi^s(T_{\mathbf{p}})} \\ &\leq b \sum_{\mathbf{p} \in \mathbf{J}} \frac{\mu(\mathcal{C}_{\mathbf{p}})^2}{\phi^s(T_{\mathbf{p}})} \\ &\leq bc_1 \sum_{k=1}^{\infty} \sum_{\mathbf{p} \in \mathbf{J}_k} \frac{\phi^t(T_{\mathbf{p}}) \mu(\mathcal{C}_{\mathbf{p}})}{\phi^s(T_{\mathbf{p}})} \\ &\leq bc_1 \sum_{k=1}^{\infty} \sum_{\mathbf{p} \in \mathbf{J}_k} \alpha_+^{k(t-s)} \mu(\mathcal{C}_{\mathbf{p}}) \\ &\leq bc_1 \mu(E) \sum_{k=1}^{\infty} \alpha_+^{k(t-s)} < \infty, \end{aligned}$$

since $\alpha_+ < 1$. We conclude that for ν -almost all $\mathbf{a} \in \mathbb{R}^{nN}$ we have $I_s(\mu_{\mathbf{a}}) < \infty$ and so $\dim_{\mathbb{H}} F(\mathbf{a}) \geq s$, contradicting that ν is supported by $E(s)$. ■

References

- [1] C. Bluhm, Fourier asymptotics of statistically self-similar measures, *J. Fourier Anal. App.*, **5** (1999), 355-362.
- [2] K. J. Falconer, Hausdorff dimension and the exceptional set of projections *Mathematika*, **29** (1982), 109-115.
- [3] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, 1985.
- [4] K. J. Falconer, The Hausdorff dimension of self-affine fractals, *Math. Proc. Cambridge Philos. Soc.*, **103** (1988), 169-179.
- [5] K. J. Falconer, The dimension of self-affine fractals II, *Math. Proc. Cambridge Philos. Soc.*, **111** (1992), 339-350.
- [6] K. J. Falconer, Bounded distortion and dimension for nonconformal repellers, *Math. Proc. Cambridge Philos. Soc.*, **115** (1994), 315-334.
- [7] K. J. Falconer, *Fractal Geometry - Mathematical Foundations and Applications*, 2nd edn., John Wiley, Chichester, 2003.
- [8] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.*, **30** (1981), 713-747.
- [9] T. Jordan, M. Pollicott and K. Simon, Hausdorff dimension for randomly perturbed self affine attractors, to appear *Comm. Math. Phys.*
- [10] J.-P. Kahane, *Some Random Series of Functions*, 2nd edn., Cambridge University Press, 1985.
- [11] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.
- [12] Y. Peres and W. Schlag, Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions, *Duke Math. J.*, **102** (2000), 193-251.
- [13] Y. Peres and B. Solomyak, Problems on self-similar and self-affine sets: an update, In *Fractal Geometry and Stochastics II, Progr. Probab.*, **102** pp 95-106, Birkhäuser, 2000.
- [14] C.A. Rogers, *Hausdorff Measures*, Cambridge University Press, 1998.
- [15] B. Solomyak, Measure and dimensions for some fractal families, *Math. Proc. Cambridge Philos. Soc.*, **124** (1998), 531-546.