The Hausdorff dimension of pulse-sum graphs

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Abstract

We consider random functions formed as sums of pulses

$$F(t) = \sum_{n=1}^{\infty} n^{-\alpha/D} G(n^{1/D}(t - X_n)) \quad (t \in \mathbb{R}^D)$$

where X_n are independent random vectors, $0 < \alpha < 1$, and G is an elementary "pulse" or "bump". Typically such functions have fractal graphs and we find the Hausdorff dimension of these graphs using a novel variant on the potential theoretic method.

1 Introduction

Many types of random fractal function have been proposed to model a wide range of phenomena from internet traffic to stock prices. One class of construction, studied in [1] and [5], depends on the superposition of randomly located "pulses" or "bumps" with width and amplitude decreasing in a self-similar manner. Here we investigate the Hausdorff dimension of the graph of such pulse-sum functions, which provides a measure of the irregularity or volatility of the process.

Let $g : \mathbb{R} \to \mathbb{R}$ be an even continuous function, decreasing on $[0, 1]$, equal to 0 on $[1, \infty)$ and such that $g(0) = 1$. We define the *elementary pulse* or *elementary bump* $G : \mathbb{R}^D \to \mathbb{R}$ to be the symmetrical function

$$G(t) = g(\|t\|)$$

where $\|t\| = \max\{|t_i|\}$ for $t = (t_1, ..., t_D) \in \mathbb{R}^D$. (The simplest instance to bear in mind is the "triangular bump" on $\mathbb{R}$, where $G(t) = g(t) = \max\{1 - |t|, 0\}$.) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we study the random function $F : \mathbb{R}^D \to \mathbb{R}$ given by a sum of randomly centred pulses

$$F(t) = \sum_{n=1}^{\infty} n^{-\alpha/D} G(n^{1/D}(t - X_n)), \tag{1}$$

where $0 < \alpha < 1$ and $(X_n)_{n\geqslant 1}$ is a sequence of independent random variables uniformly distributed on $[-1, 2]^D$. Note that these random functions are stationary (restricted to $[0, 1]^D$) and locally self-affine.

Such pulse sums were introduced in one dimension in [5], but with bumps distributed according to a Poisson process rather than uniformly, where their local asymptotic form and their relationship to other processes such as fractional Brownian motion and Lévy processes was discussed, see also [6]. Pulse sums of the form (1) were used as a model for rough surfaces and profiles in [9]. Their mathematical properties were analysed in more detail in [1] where it was shown that almost surely (1) defines a continuous function on $[0, 1]^D$. Moreover, if g is Hölder continuous with exponent γ then almost surely F is Hölder continuous with exponent β on $[0, 1]^D$ for all $0 < \beta < \min\{\alpha, \gamma\}$. (Recall that h is Hölder continuous with exponent γ on a domain $A \subset \mathbb{R}^D$ if

$$|h(t) - h(s)| \leqslant C \|t - s\|^\gamma \qquad \text{for all } s, t \in A.) \tag{2}$$

The graph Γ_F of F is defined by $\Gamma_F = \{ (t, F(t)) : t \in [0, 1]^D \} \subset \mathbb{R}^{D+1}$ where we identify $\mathbb{R}^D \times \mathbb{R}$ with $\mathbb{R}^{D+1}$ in the obvious way. The Hölder estimates on F immediately give an upper bound for the (upper) box-counting dimension $\dim_B(\Gamma_F)$ of the graph: almost surely

$$\dim_B(\Gamma_F) \leqslant D + 1 - \min\{\alpha, \gamma\}. \tag{3}$$

Furthermore, if $\alpha < \gamma$, and in particular if g is Lipschitz (i.e. with Hölder exponent $\gamma = 1$), we get equality

$$\dim_B(\Gamma_F) = D + 1 - \alpha. \tag{4}$$

Determination of the Hausdorff dimension of the graph, $\dim_H(\Gamma_F)$, was left as an open question in [1]. Here we obtain an almost sure lower bound for $\dim_H(\Gamma_F)$ which gives the exact value for reasonable (i.e. not too irregular) elementary pulses G.

Theorem 1.1. *Assume that there exists a non-empty interval $I \subset [0, 1]$ on which $g : I \to J$ is a $\mathcal{C}^1$–diffeomorphism. Then the Hausdorff dimension of Γ_F, the graph of the random function $F : [0, 1]^D \to \mathbb{R}$ given by (1), satisfies*

$$\dim_H(\Gamma_F) \geqslant D + 1 - \alpha$$

almost surely.

(Recall that g is a $\mathcal{C}^1-$diffeomorphism if it is continuously differentiable and invertible with g^{-1} continuously differentiable.)

Since the Hausdorff dimension of a set is never more than its box dimension, we have the following corollary.

Corollary 1.2. *Suppose g satisfies the conditions of Theorem 1.1 and is Hölder continuous with exponent γ. Then*

$$D + 1 - \alpha \leqslant \dim_H(\Gamma_F) \leqslant \dim_B(\Gamma_F) \leqslant D + 1 - \min\{\alpha, \gamma\}$$

almost surely. In particular we have equality if $\alpha \leqslant \gamma$ which includes the case when g is a Lipschitz function.

Our proof of Theorem 1.1 is based on the potential theoretic method which has been used to obtain the Hausdorff dimension of graphs of many functions, such as fractional Brownian motion [2],[4] and the random Weierstrass function [3],[8]. However, this method cannot be used directly for sums of pulses, where we first require careful conditioning to construct random subsets of $[0,1]^D$ on which appropriate measures may be defined. The potential theoretic ideas are developed in Section 2 leading to the main proof in Section 3 with some generalisations indicated in Section 4.

2 Dimension and energy

Recall that the *s-dimensional Hausdorff measure* of $K \subset \mathbb{R}^{D+1}$ is defined by

$$\mathcal{H}^s(K) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(K) = \sup_{\varepsilon > 0} \mathcal{H}^s_\varepsilon(K)$$

where, for $\varepsilon > 0$,

$$\mathcal{H}^s_\varepsilon(K) = \inf \left\{ \sum_{i=0}^{\infty} |E_i|^s \ : E \subset \bigcup_{i=0}^{\infty} E_i \text{ and } |E_i| \leqslant \varepsilon \right\},$$

with $|A|$ denoting the diameter of a set $A \subset \mathbb{R}^{D+1}$. The *Hausdorff dimension* of K is then given by

$$\dim_H(K) = \inf\{s \ : \ \mathcal{H}^s(K) = 0\} = \sup\{s \ : \ \mathcal{H}^s(K) = \infty\},$$

see [2],[4]. When calculating Hausdorff dimensions, the difficulty is often to find lower estimates for $\dim_H(K)$, and one approach is to relate Hausdorff dimension to certain energy integrals.

Given a finite non-null Radon measure μ we define the *s-energy* of μ by

$$I_s(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{\|x-y\|^s}. \tag{5}$$

Then

$$\dim_H(K) = \sup\{s \geqslant 0 \,:\, I_s(\mu) < \infty \text{ for some finite } \mu \text{ supported by } K\},$$

see [2],[4]; in particular if we can construct a measure μ supported by K with finite s-energy then $\dim_H(K) \geq s$. For the graph $\Gamma_F = \{\,(t, F(t)) \,:\, t \in [0,1]^D\,\} \subset \mathbb{R}^{D+1}$ of a continuous function $F : [0,1]^D \to \mathbb{R}$, there is a natural measure μ on Γ_F obtained by lifting D-dimensional Lebesgue measure λ on $[0,1]^D$ onto Γ_F. Formally,

$$\mu(E) = \lambda\{t \in [0,1]^D \text{ such that } (t, F(t)) \in E\} \quad \text{for all } E \subset \mathbb{R}^{D+1}.$$

It is convenient to define the "Euclidean" norm on $\mathbb{R}^{D+1} \equiv \mathbb{R}^D \times \mathbb{R}$ by

$$\|x\|_2 = (\|t\|^2 + |u|^2)^{1/2} \quad \text{for all } x \equiv (t,u) \in \mathbb{R}^{D+1}.$$

Since the norms are equivalent, we can redefine (5) in terms of $\|.\|_2$, to get

$$I_s(\mu) = \iint_{[0,1]^D \times [0,1]^D} (|F(x) - F(y)|^2 + \|x-y\|^2)^{-s/2}\, dx\, dy. \tag{6}$$

Note that the finiteness of $I_s(\mu)$ depends on the relative size of the increments $|F(x) - F(y)|$ when $\|x - y\|$ is small.

Now suppose that F is a random process with continuous sample paths. To show that the integral (6) is finite almost surely it is enough to show that its expectation is finite, and by Fubini's theorem this will follow from a suitable bound on

$$\mathbb{E}((|F(x) - F(y)|^2 + \|x-y\|^2)^{-s/2}). \tag{7}$$

For fixed $x, y \in [0,1]^D$ we define a random variable $Z = F(x) - F(y)$. If we can bound the distribution function $p(r) = \mathbb{P}(|Z| < r)$ of $|Z|$ then we may be able to bound (7) and thus (6). In particular, if Z admits a density f_Z we get a simple criterion for convergence: if f_Z is bounded on $\mathbb{R}$ and there exists a constant $C > 0$, independent of x, y, such that

$$\|f_Z\|_\infty \leqslant C\, \|x-y\|^{-\alpha} \quad (\alpha > 0),$$

and $s < D + 1 - \alpha$ then $\mathbb{E}(I_s(\mu)) < \infty$. (For example, this approach easily gives $\dim_H(\Gamma) \geqslant 2 - \alpha$ when F is fractional Brownian motion of index α, see [2]).

For our pulse sums, we have for fixed $x, y \in [0,1]^D$

$$Z = F(x) - F(y) = \sum_{n=1}^{\infty} Z_n$$

where

$$Z_n = n^{-\alpha/D}(G(n^{1/D}(x - X_n)) - G(n^{1/D}(y - X_n))).$$

The main problem is that the pulses have narrow support so the random variables Z_n have a highly non-uniform density when n is large, rendering a useful estimate of (7) difficult. We circumvent this problem by careful conditioning. Specifically, we consider the random sets

$$I_n = \{t \in \mathbb{R}^D \,:\, \|n^{1/D}(t - X_n)\| \in I\}, \tag{8}$$

where I is the interval on which g is a diffeomorphism. We then define random sets

$$V_k = I_{m_k} \cup \ldots \cup I_{m_{k+1}-1}$$

where $m_k = 2^{k^2}$. For fixed $x, y \in [0,1]^D$ with $\|x - y\| \approx m_k^{-1}$, we bound

$$\mathbb{P}((|F(x) - F(y)| < r) \cap (x \in V_k))$$

and this gives an adequate estimate of

$$\mathbb{E}((|F(x) - F(y)|^2 + \|x - y\|^2)^{-s/2}\mathbb{1}_{(x \in V_k)}),$$

where $\mathbb{1}$ denotes the indicator function. For a suitably large k_0, we define the random set

$$W = [0,1]^D \cap \big(\bigcap_{k \geqslant k_0} V_k\big).$$

Writing λ_W for the restriction of Lebesgue measure to W, we show that for all $1 < s < D + 1 - \alpha$:

$$\mathbb{E}\left(\iint_{[0,1]^D \times [0,1]^D} \big((|F(x) - F(y)|^2 + \|x - y\|^2)^{-s/2}\big)\, d\lambda_W(x)\, d\lambda_W(y)\right) < \infty,$$

and also show that $\lambda(W) > 0$ with probability arbitrarily close to 1. The result then follows from the energy criterion on lifting the measure λ_W onto the graph Γ_F.

3 Proof of the main theorem

We proceed by a sequence of intermediate results. Our first aim, which we achieve in Corollary 3.3, is to bound the probability that the increment $|F(x) - F(y)|$ is small under suitable conditioning.

We use the following probability notation. For each event $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ we write $\mathbb{P}^A$ for probability conditional on A, so that

$$\mathbb{P}^A(B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \quad \text{for all } B \in \mathcal{F}.$$

Then $\mathbb{P}^A$ is absolutely continuous with respect to $\mathbb{P}$ with density

$$\frac{d\mathbb{P}^A}{d\mathbb{P}} = \frac{1}{\mathbb{P}(A)} 1\!\!1_A.$$

We write $\mathbb{E}^A$ for expectation with respect to $\mathbb{P}^A$, so that for all random variables X,

$$\mathbb{E}^A(X) = \frac{1}{\mathbb{P}(A)} \mathbb{E}(X 1\!\!1_A).$$

Finally, we let $\mathbb{P}_X$ denote the law of X as a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, so

$$\mathbb{P}_X(B) = \mathbb{P}(X \in B) \quad \text{for all Borel sets } B.$$

For fixed $x, y \in [0,1]^D$ we define, as before,

$$Z = F(x) - F(y) = \sum_{n=1}^{\infty} Z_n$$

where

$$Z_n = n^{-\alpha/D}(G(n^{1/D}(x - X_n)) - G(n^{1/D}(y - X_n)))$$

For this fixed x, we write A_n for the event $(x \in I_n)$, where $I_n = \{t \in \mathbb{R}^D : \left\|n^{1/D}(t - X_n)\right\| \in I\}$, so that $A_n = (\left\|n^{1/D}(x - X_n)\right\| \in I)$.

Lemma 3.1. *Let $x, y \in [0,1]^D$ be given. For all $p \geqslant 1$ such that $\|x - y\| > 2p^{-1/D}$, the random variable Z_p has a density conditional on A_p given by*

$$f_p(z) = C_D \frac{p^{-1+\alpha/D}}{\mathbb{P}(A_p)} (h(p^{\alpha/D} z))^{D-1} \left|h'(p^{\alpha/D} z)\right| 1\!\!1_J(p^{\alpha/D} z) \quad \text{for all } z \in \mathbb{R},$$

where $h : J \to I$ is the inverse of g and $C_D > 0$ is a constant only depending on D.

Proof. For all positive functions $\Phi : \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}^{A_p}(\Phi(Z_p)) = \int_\Omega \Phi(Z_p)\, d\mathbb{P}^{A_p} = \frac{1}{\mathbb{P}(A_p)} \int_\Omega \Phi(Z_p) \mathbb{1}_{A_p}\, d\mathbb{P} = \frac{1}{\mathbb{P}(A_p)} \int_\Omega \Phi(Z_p) \mathbb{1}_{I_p}(x)\, d\mathbb{P}.$$

Since $\|x-y\| > 2p^{-1/D}$, the points x and y cannot both lie in the support of $t \mapsto G(p^{1/D}(t - X_p))$. This set contains I_p so $Z_p = p^{-\alpha/D} G(p^{1/D}(x - X_p))$ whenever $x \in I_p$. Thus

$$\begin{aligned} \mathbb{E}^{A_p}(\Phi(Z_p)) &= \frac{1}{\mathbb{P}(A_p)} \int_\Omega \Phi(p^{-\alpha/D} G(p^{1/D}(x - X_p))) \mathbb{1}_I(\|p^{1/D}(x - X_p)\|)\, d\mathbb{P} \\ &= \frac{1}{\mathbb{P}(A_p)} \int_{\mathbb{R}^D} \Phi(p^{-\alpha/D} g(\|p^{1/D}(x-u)\|)) \mathbb{1}_I(\|p^{1/D}(x-u)\|)\, f(u)\, du \\ &= \frac{p^{-1}}{\mathbb{P}(A_p)} \int_{\mathbb{R}^D} \Phi(p^{-\alpha/D} g(\|v\|)) \mathbb{1}_I(\|v\|)\, f(x - p^{-1/D} v)\, dv \end{aligned}$$

where $f(u) = 3^{-D} \mathbb{1}_{[-1,2]^D}(u)$ is the density of X_p. For all $x \in [0,1]^D$, all $p \geqslant 1$ and all $v \in \mathbb{R}^D$, if $\|v\| \in I$ then $x - p^{-1/D} v \in [-1,2]^D$. Thus

$$\begin{aligned} \mathbb{E}^{A_p}(\Phi(Z_p)) &= \frac{p^{-1}}{3^D\, \mathbb{P}(A_p)} \int_{\mathbb{R}^D} \Phi(p^{-\alpha/D} g(\|v\|)).\mathbb{1}_I(\|v\|)\, dv \\ &= \frac{D 2^D p^{-1}}{3^D\, \mathbb{P}(A_p)} \int_{\mathbb{R}^+} \Phi(p^{-\alpha/D} g(r)) \mathbb{1}_I(r)\, r^{D-1}\, dr, \end{aligned}$$

by the formula for integrating a 'radial' function, noting that $\lambda\{x : \|x\| \le 1\} = 2^D$. Thus, setting $z = p^{-\alpha/D} g(r)$,

$$\begin{aligned} \mathbb{E}^{A_p}(\Phi(Z_p)) &= \frac{D2^D}{3^D} \frac{p^{-1+\alpha/D}}{\mathbb{P}(A_p)} \int_{\mathbb{R}^+} \Phi(z)\, \mathbb{1}_I(h(p^{\alpha/D} z))\, (h(p^{\alpha/D} z))^{D-1} \left|h'(p^{\alpha/D} z)\right| dz \\ &= \frac{D2^D}{3^D} \frac{p^{-1+\alpha/D}}{\mathbb{P}(A_p)} \int_{\mathbb{R}^+} \Phi(z)\, \mathbb{1}_J(p^{\alpha/D} z)\, (h(p^{\alpha/D} z))^{D-1} \left|h'(p^{\alpha/D} z)\right| dz, \end{aligned}$$

giving the result with $C_D = D(2/3)^D$.

We next isolate Z_p in the sum defining Z. Let $S_p = \sum_{n \neq p} Z_n$ so that $Z = S_p + Z_p$. We now condition on S_p.

Corollary 3.2. *Let $x, y \in [0,1]^D$ be given and $p \geqslant 1$ be such that $\|x-y\| > 2p^{-1/D}$. Then Z regarded as a random variable on $(\Omega, \mathcal{F}, \mathbb{P}^{A_p})$ has a density conditional on S_p given by*

$$f_Z^{S_p = s}(z) = f_p(z - s) \quad \text{for all } z \in \mathbb{R}, \tag{9}$$

where f_p is as in Lemma 3.1.

Proof. We find the law of the pair (S_p, Z). For all Borel A, B we have

$$\mathbb{P}^{A_p}_{(S_p,Z)}(A,B) = \mathbb{P}^{A_p}_{\varphi(S_p,Z_p)}(A,B)$$

where $\varphi(s,z) = (s, s+z)$ is a $\mathcal{C}^1-$diffeomorphism with Jacobian 1, with $\varphi^{-1}(s,z) = (s, z-s)$. Thus

$$\mathbb{P}^{A_p}_{(S_p,Z)}(A,B) = \mathbb{P}^{A_p}_{(S_p,Z_p)}(A, B-A) = \mathbb{P}^{A_p}_{S_p}(A)\,\mathbb{P}^{A_p}_{Z_p}(B-A)$$

since S_p and Z_p are independent. From Lemma 3.1

$$\begin{aligned}
\mathbb{P}^{A_p}_{(S_p,Z)}(A,B) &= \int_A d\mathbb{P}^{A_p}_{S_p}(s) \int_{B-A} d\mathbb{P}^{A_p}_{Z_p}(z) \\
&= \int_A \int_{B-A} d\mathbb{P}^{A_p}_{Z_p}(z)\, d\mathbb{P}^{A_p}_{S_p}(s) \\
&= \int_A \int_{B-A} f_p(z)\, dz\, d\mathbb{P}^{A_p}_{S_p}(s) \\
&= \int_A \left(\int_B f_p(z-s)\, dz \right) d\mathbb{P}^{A_p}_{S_p}(s).
\end{aligned}$$

Thus Z has a density conditional on S_p given by (9).

Corollary 3.3. *For all $n > m$, all $x, y \in [0,1]^D$ such that $\|x-y\| > 2\,m^{-1/D}$ and all $r > 0$,*

$$\mathbb{P}((|F(x) - F(y)| < r) \cap (A_m \cup \ldots \cup A_n)) \leqslant Crn^{\alpha/D}$$

for some $C > 0$.

Proof. Let $p \geqslant m$. From Lemma 3.1 and Corollary 3.2, for all $s \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}((|Z| < r) \cap A_p \,|\, S_p = s) &= \mathbb{P}((|Z| < r) \,|\, A_p \,\big|\, S_p = s)\,\mathbb{P}(A_p) \\
&= \left(\int_{[-r,r]} f_p(z-s)\, dz \right) \mathbb{P}(A_p) \\
&\leqslant C_D \frac{p^{-1+\alpha/D}}{\mathbb{P}(A_p)} \|h\|_\infty^{D-1} \|h'\|_\infty (2\,r)\,\mathbb{P}(A_p) \\
&\leqslant Crp^{-1+\alpha/D}.
\end{aligned}$$

Hence

$$\mathbb{P}((|Z| < r) \cap A_p) = \int_{\mathbb{R}} \mathbb{P}((|Z| < r) \cap A_p \,|\, S_p = s)\, d\mathbb{P}_{S_p}(s) \leqslant Crp^{-1+\alpha/D}.$$

Thus

$$\begin{aligned}\mathbb{P}((|Z|<r)\cap(A_m\cup\ldots\cup A_n)) &\leqslant \sum_{p=m}^{n}\mathbb{P}((|Z|<r)\cap A_p)\\ &\leqslant \sum_{p=m}^{n} Crp^{-1+\alpha/D} \leqslant C' r n^{\alpha/D}.\end{aligned}$$

We use Corollary 3.3 to obtain a corresponding estimate for the expectations.

Corollary 3.4. *Let $s>1$. For $1\leqslant m<n$ let V be the random set $V=I_m\cup\ldots\cup I_n$. Let $x,y\in[0,1]^D$ be such that $\|x-y\|>2\,m^{-1/D}$. Then*

$$\mathbb{E}((|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2}\,\mathbb{1}_{(x\in V)})\leqslant C\,\|x-y\|^{1-s}\,n^{\alpha/D} \tag{10}$$

for some $C>0$.

Proof. Set $h=\|x-y\|$ and, for $r>0$, write

$$p(r)\equiv\mathbb{P}((|Z|<r)\cap(x\in V))\leqslant Crn^{\alpha/D}$$

by Corollary 3.3. Then

$$\begin{aligned}\mathbb{E}^{(x\in V)}((|Z|^2+h^2)^{-s/2}) &= \int_0^\infty (r^2+h^2)^{-s/2}\,d(\mathbb{P}^{(x\in V)}(|Z|<r))\\ &= \frac{1}{\mathbb{P}(x\in V)}\int_0^\infty (r^2+h^2)^{-s/2}\,dp(r).\end{aligned}$$

Hence

$$\mathbb{E}((|Z|^2+h^2)^{-s/2}\mathbb{1}_{(x\in V)})=\int_0^\infty (r^2+h^2)^{-s/2}\,dp(r).$$

Using integration by parts, we get, for appropriate constants C,

$$\begin{aligned}\int_0^\infty (r^2+h^2)^{-s/2}\,dp(r) &\leqslant \int_0^h h^{-s}\,dp(r)\;+\;\int_h^\infty r^{-s}\,dp(r)\\ &\leqslant h^{-s}p(h)\;+\;[r^{-s}p(r)]_{r=h}^\infty\;+\;s\int_h^\infty r^{-s-1}p(r)\,dr\\ &\leqslant Ch^{-s}hn^{\alpha/D}\;+\;C\int_h^\infty r^{-s-1}rn^{\alpha/D}\,dr\\ &\leqslant Ch^{1-s}n^{\alpha/D}\;+\;Cn^{\alpha/D}[-r^{1-s}]_h^\infty\\ &\leqslant Ch^{1-s}n^{\alpha/D}\end{aligned}$$

giving (10).

Our next aim is essentially to show that, for given x, y, the increment $|F(x) - F(y)|$ has a high probability of being suitably large, for x in a (large) random subset of $[0,1]^D$.

Define an increasing sequence of integers by $m_k = 2^{k^2}$ and define the random set $V_k = I_{m_k} \cup \ldots \cup I_{m_{k+1}-1}$. For all $\varepsilon > 0$ let k_ε be the least positive integer such that for all $k \geqslant k_\varepsilon$

$$\left(\frac{k+1}{k-1}\right)^2 = \left(1 + \frac{2}{k-1}\right)^2 \leqslant 1 + \varepsilon,$$

and set $h_\varepsilon = 2\, m_{k_\varepsilon - 1}^{-1/D}$.

Corollary 3.5. *Let $s > 1$ and $\varepsilon > 0$. Given $x, y \in [0,1]^D$ such that $\|x - y\| < h_\varepsilon$, let $k \geqslant 1$ be the unique integer satisfying*

$$\frac{2}{m_k^{1/D}} < \|x - y\| \leqslant \frac{2}{m_{k-1}^{1/D}}.$$

Then

$$\mathbb{E}((|F(x) - F(y)|^2 + \|x - y\|^2)^{-s/2} \mathbb{1}_{(x \in V_k)}) \leqslant C \|x - y\|^{1-s-(1+\varepsilon)\alpha}$$

for some $C > 0$.

Proof. Since $\|x - y\| < h_\varepsilon$ we have $k \geqslant k_\varepsilon$ where k_ε is as above. Thus

$$\begin{aligned} m_{k+1} = m_{k-1}^{((k+1)/(k-1))^2} &\leqslant 2^{D((k+1)/(k-1))^2} \|x - y\|^{-D((k+1)/(k-1))^2} \\ &\leqslant C \|x - y\|^{-D(1+\varepsilon)} \end{aligned}$$

using the definitions of m_k and k_ε. Applying Collorary 3.4 with $m = m_k$, $n = m_{k+1} - 1$ and $V = V_k$,

$$\begin{aligned} \mathbb{E}((|F(x) - F(y)|^2 + \|x - y\|^2)^{-s/2} \mathbb{1}_{(x \in V_k)}) &\leqslant C \|x - y\|^{1-s} m_{k+1}^{\alpha/D} \\ &\leqslant C' \|x - y\|^{1-s} (\|x - y\|^{-D(1+\varepsilon)})^{\alpha/D} \\ &\leqslant C' \|x - y\|^{1-s-(1+\varepsilon)\alpha}. \end{aligned}$$

The next lemma will enable us to estimate the measure of the V_k.

Lemma 3.6. *There exists a constant $\delta > 0$ such that for all $1 \leqslant m < n$:*

$$\mathbb{E}\left(\lambda\left([0,1]^D \setminus \bigcup_{p=m}^{n} I_p\right)\right) \leqslant \left(\frac{m}{n}\right)^\delta$$

Proof. Since $I_p \subset [-1,2]^D$ for all $p \geqslant 1$, we have

$$\mathbb{P}(x \in I_p) = 3^{-D}\lambda(I_p) = \frac{3^{-D}\,|I|^D}{p} \equiv \frac{C}{p},$$

say. Then

$$\mathbb{E}\left(\lambda\left([0,1]^D \setminus \bigcup_{p=m}^{n} I_p\right)\right) = \mathbb{E}\left(\int_{[0,1]^D} \mathbb{I}_{\{x \notin \cup_m^n I_p\}}\, d\lambda\right)$$
$$= \int_{[0,1]^D} \mathbb{E}(\mathbb{I}_{\{x \notin \cup_m^n I_p\}})\, d\lambda = \prod_{p=m}^{n}\left(1 - \frac{C}{p}\right).$$

But

$$\log \prod_{p=m}^{n}\left(1 - \frac{C}{p}\right) = \sum_{p=m}^{n} \log\left(1 - \frac{C}{p}\right) \leqslant -\sum_{p=m}^{n} \frac{C}{p} \leqslant -\delta \int_m^n \frac{dt}{t} = \delta \log\left(\frac{m}{n}\right)$$

for some constant $\delta > 0$, and the conclusion follows.

We now prove our main theorem.

Proof of Theorem 1.1. Fix $1 < s < D + 1 - \alpha$ and $0 < \eta < 1$. Take $\varepsilon > 0$ such that $(1+\varepsilon)\alpha < D + 1 - s < D$ with $k_\varepsilon \geqslant 1$ the associated integer. Let $k_0 \geqslant k_\varepsilon$ and $h_0 = 2/m_{k_0-1}^{1/D}$. We define the random set

$$W = [0,1]^D \cap \Big(\bigcap_{k=k_0}^{\infty} V_k\Big)$$

and let λ_W denote the restriction of Lebesgue measure to W.

To prove the theorem we first show that the measure obtained by lifting λ_W onto Γ_F has finite s-energy and then that this measure is positive.

(a) For convenience, write

$$R_k = \{\, (x,y) \in [0,1]^D \times [0,1]^D \;:\; 2/m_k^{1/D} < \|x - y\| \leqslant 2/m_{k-1}^{1/D} \,\}.$$

Noting that by definition $W \subset V_k$ for all $k \geqslant k_0$,

$$
\begin{aligned}
&\mathbb{E}\left(\iint_{\{x,y\in[0,1]^D,\,\|x-y\|\leqslant h_0\}} (|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2} d\lambda_W(x)\, d\lambda_W(y)\right)\\
&\leqslant \mathbb{E}\left(\iint_{\{x\in W,\, y\in[0,1]^D,\,\|x-y\|\leqslant h_0\}} (|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2} dx\, dy\right)\\
&\leqslant \mathbb{E}\left(\sum_{k=k_0}^{\infty}\iint_{R_k\bigcap W\times[0,1]^D} (|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2} dx\, dy\right)\\
&\leqslant \mathbb{E}\left(\sum_{k=k_0}^{\infty}\iint_{R_k} (|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2}\, \mathbb{1}_{(x\in V_k)}\, dx\, dy\right)\\
&\leqslant \sum_{k=k_0}^{\infty}\left(\iint_{R_k} \mathbb{E}((|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2}\, \mathbb{1}_{(x\in V_k)})\, dx\, dy\right)\\
&\leqslant C\sum_{k=k_0}^{\infty}\left(\iint_{R_k} \|x-y\|^{1-s-(1+\varepsilon)\alpha}\; dx\, dy\right)\\
&\leqslant C\iint_{\{x,y\in[0,1]^D,\,\|x-y\|\leqslant h_0\}} \|x-y\|^{1-s-(1+\varepsilon)\alpha}\; dx\, dy,
\end{aligned}
$$

using Corollary 3.5. Since $1-s-(1+\varepsilon)\alpha > -D$, this last integral converges, so

$$
\iint_{\{x,y\in[0,1]^D,\,\|x-y\|\leqslant h_0\}} (|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2} d\lambda_W(x)\, d\lambda_W(y) < \infty
$$

almost surely. Since the integral is finite on the complementary domain of integration $R_{k_0}^c = \{x,y\in[0,1]^D \,:\, \|x-y\| > h_0\}$, we have

$$
\iint_{[0,1]^D\times[0,1]^D} (|F(x)-F(y)|^2+\|x-y\|^2)^{-s/2} d\lambda_W(x)\, d\lambda_W(y) < \infty
$$

almost surely. If μ_W denotes the lift of λ_W onto Γ_F, so that $\mu_W(E) = \lambda_W\{t \in [0,1]^D$ such that $(t, F(t)) \in E\}$ for all $E \in \mathbb{R}^{D+1}$, it follows that μ_W is a measure of finite s-energy that is supported by Γ_F. Thus to conclude that $\dim_H(\Gamma_F) \geqslant s$ it remains to show that μ_W is positive or, equivalently, that $\lambda(W) > 0$.

(b) We have

$$
[0,1]^D \backslash W = \bigcup_{k=k_0}^{\infty}\left([0,1]^D \backslash \bigcup_{p=m_k}^{m_{k+1}-1} I_p\right)
$$

so by Lemma 3.6

$$\mathbb{E}(\lambda(\,[0,1]^D\backslash W\,)) \leqslant \sum_{k=k_0}^{\infty} \left(\frac{m_k}{m_{k+1}-1}\right)^{\delta}.$$

But for $k \geqslant 1$

$$\frac{m_k}{m_{k+1}-1} = \frac{2^{\,k^2}}{2^{\,(k+1)^2}-1} \leqslant 2.2^{\,k^2-(k+1)^2} = 2^{-\,2k}$$

so that

$$\mathbb{E}(\lambda(\,[0,1]^D\backslash W\,)) \leqslant \sum_{k=k_0}^{\infty} 2^{-\,2k\delta} = \frac{2^{\,-2k_0\delta}}{1-2^{\,-2\delta}}.$$

By Markov's inequality

$$\mathbb{P}(\lambda(W) < 1/2) = \mathbb{P}(\lambda(\,[0,1]^D\backslash W\,) \geqslant 1/2) \leqslant 2\,\frac{2^{\,-2k_0\delta}}{1-2^{\,-2\delta}}.$$

If we choose k_0 large enough, then $(2^{\,1-2k_0\delta})/(1-2^{\,-2\delta}) < \eta$, so $\lambda(W) \geqslant 1/2$ with probability greater than $1-\eta$.

(c) From (a) and (b) we see that $\dim_H(\Gamma_F) \geqslant s$ with probability at least $1-\eta$. Since $1 < s < D+1-\alpha$ and $0 < \eta < 1$ are arbitrary, we conclude that $\dim_H(\Gamma_F) \geqslant D+1-\alpha$ almost surely.

4 Variations and generalisations

There are many variants of the basic "pulse" construction to which these methods may be applied.

Firstly, the analysis extends to random functions defined by elementary pulses G of a much more general form. For example, if in (1) we take $G(t) = g(\|t\|_G)$ where $\|\|_G$ is *any* norm such that its unit ball $\{t : \|t\|_G \leq 1\}$ is contained in $[-1,1]^D$ then similar results hold.

These processes are also of interest for more general probability distributions ν of the random variables X_n. For the random function F to exist, the pulses $G(n^{1/D}(t - X_n))$ must not accumulate at any point. On the other hand, it is the stackings of the pulses that result in the iregularities in the signal. These considerations lead to natural assumptions on the law of ν.

Let ν be a Borel probability measure on $\mathbb{R}^D$ such that its density with respect to Lebesgue measure exists and is bounded above, and such that there is a ball B on which this density is bounded away from 0. Then the arguments in [1] show that the series (1) almost surely defines a continuous function on B with the box dimension of the graph satisfying (3). The box dimension of Γ_F is given by (4) under a further hypothesis on ν.

The real case, $D = 1$, where the pulses may be thought of as "signals" of some form, is of particular interest. Under these conditions on ν the conclusions of Theorem 1.1 remain valid. For this, Lemma 3.1 easily generalises and the upper bound on the density allows us to bound $\|f_p\|_\infty$ in Corollary 3.3. Finally, the lower bound on the density means that $\mathbb{P}(x \in I_p)$ is not too small in Lemma 3.6.

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