Random subsets of self-affine fractals

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Abstract

We find the almost sure Hausdorff and box-counting dimensions of random subsets of self-affine fractals obtained by selecting subsets at each stage of the hierarchical construction in a statistically self-similar manner.

1 Introduction

An iterated function system (IFS) consists of a finite family of contractions $S_1, \ldots, S_N : \mathbb{R}^n \to \mathbb{R}^n$ with $N \geq 2$. It is well-known that this defines a unique non-empty compact set $F \subseteq \mathbb{R}^n$ such that

$$F = \bigcup_{i=1}^{N} S_i(F), \tag{1.1}$$

called the *attractor* of the IFS, see, for example, [7, 11]. If the S_i are similarities we call F *self-similar* and if they are affine mappings we call F *self-affine*.

The attractor F may be obtained by an iterative procedure. We index compositions of the S_i in the usual way by sequences $\mathbf{i} = (i_1 \ldots i_k)$, where $i_j \in \{1, \ldots, N\}$, so that $S_{\mathbf{i}} \equiv S_{i_1} \circ \ldots \circ S_{i_k}$, etc, and we write $|\mathbf{i}|$ for the length of the sequence $\mathbf{i}$. If B is some (large) ball or other compact set such that $S_i(B) \subseteq B$ for all i then the sequence of compact sets $F_k \equiv \cup_{|\mathbf{i}|=k} S_{\mathbf{i}}(B)$ decreases to F as $k \to \infty$. The set F_{k+1} may be thought of as being obtained from F_k by replacing each of its components $S_{\mathbf{i}}(B)$ by the N subsets $S_{\mathbf{i}}S_1(B), \ldots, S_{\mathbf{i}}S_N(B)$. In this paper we instead consider the random compact subsets of F obtained by replacing each $S_{\mathbf{i}}(B)$ by a *random* subcollection of the $S_{\mathbf{i}}S_1(B), \ldots, S_{\mathbf{i}}S_N(B)$. If the probability distribution of the choice of the subcollection indices $\{1, \ldots, N\}$ is the same at each stage, this construction might be regarded as a particular example of a *statistically self-affine* set.

A great deal is known about the Hausdorff and box-counting dimensions of self-similar sets, see for example [7, 11], and also of their natural random counterparts [4, 10, 15]. Dimensions of self-affine sets are more awkward to analyse,[2, 5, 6, 11, 14]. Very little has been done to extend this theory to the random setting: working in a submultiplicative rather than a multiplicative setting presents many challenges. Gatzouras and Lalley [9] consider a specific random construction based on the generalized Sierpiński carpets, Jordan, Pollicott and Simon [13] look at random perturbations of self-affine constructions but keeping the linear part of the affine maps fixed, and Jinghu and Lixin [12] work in a setting that is close to self-similar and so avoids the main difficulties. Here we investigate the random variant indicated above, basing our treatment on random subtrees of the N-ary rooted tree that underlies the IFS construction.

2 Description of the model

In this section we describe the random construction in detail.

We code the IFS construction in the usual way. For each $k = 0, 1, 2, \ldots$ let $\mathbf{J}_k = \{(i_1 \ldots i_k) : 1 \leq i_j \leq N\}$ be the set of sequences or words of length k, with $\mathbf{J}_0$ containing only the null sequence $\emptyset$. Let $\mathbf{J} = \bigcup_{k=0}^{\infty} \mathbf{J}_k$ be the set of all finite sequences, and let $\mathbf{J}_\infty = \{(i_1 i_2 \ldots) : 1 \leq i_j \leq N\}$ be the corresponding set of infinite sequences. We abbreviate members of $\mathbf{J}$ and $\mathbf{J}_\infty$ as $\mathbf{i} = (i_1 \ldots i_k)$ and $\mathbf{i} = (i_1 i_2 \ldots)$, and denote the number of terms in $\mathbf{i} \in \mathbf{J}$ by $|\mathbf{i}|$. If $\mathbf{i}, \mathbf{j} \in \mathbf{J}$ or if $\mathbf{i} \in \mathbf{J}$ and $\mathbf{j} \in \mathbf{J}_\infty$, we denote by $\mathbf{ij}$ the sequence obtained by juxtaposition of the terms of $\mathbf{i}$ and $\mathbf{j}$. If $\mathbf{i}$ is a curtailment of $\mathbf{j}$, that is if $\mathbf{j} = \mathbf{i}\mathbf{i}'$ for some $\mathbf{i}'$, we write $\mathbf{i} \preceq \mathbf{j}$. If $\mathbf{i}, \mathbf{j} \in \mathbf{J}$ or $\mathbf{J}_\infty$, then $\mathbf{i} \wedge \mathbf{j} \in \mathbf{J}$ denotes the maximal common initial subsequence of $\mathbf{i}$ and $\mathbf{j}$. We write $\mathbf{i}|_k = i_1 \ldots i_k$ for any $\mathbf{i} \in \mathbf{J}$ with $k \leq |\mathbf{i}|$ or $\mathbf{i} \in \mathbf{J}_\infty$.

We may regard $\mathbf{J}$ as an N-ary tree rooted at $\emptyset$ in the obvious way with each vertex $\mathbf{i}$ joined to N 'children' $\{\mathbf{i}1, \ldots, \mathbf{i}N\}$. Then $\mathbf{J}_\infty$, which may be thought of as the boundary of $\mathbf{J}$, may be made into a compact metric space using the metric $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$ for distinct $\mathbf{i}, \mathbf{j} \in \mathbf{J}_\infty$. The *cylinders* $\mathcal{C}_\mathbf{i} = \{\mathbf{j} \in \mathbf{J}_\infty : \mathbf{i} \preceq \mathbf{j}\}$ for $\mathbf{i} \in \mathbf{J}$ form a base of open and closed neighborhoods for $\mathbf{J}_\infty$.

We call $\mathbf{T} \subseteq \mathbf{J}$ a *tree* or a *subtree* of $\mathbf{J}$ if for all $\mathbf{i} \in \mathbf{T}$, we have $\mathbf{i}|_k \in \mathbf{T}$ for all $0 \leq k \leq |\mathbf{i}|$. Given a tree $\mathbf{T}$, we write $\mathbf{T}_k = \{\mathbf{i} \in \mathbf{T} : |\mathbf{i}| = k\}$ for its kth level vertices and $\mathbf{T}_\infty = \{\mathbf{i} \in \mathbf{J}_\infty : \mathbf{i}|_k \in \mathbf{T} \text{ for all } k\} \subseteq \mathbf{J}_\infty$ for its boundary, which we may think of as the limiting infinite tree.

Points of IFS attractors may be coded using this notation. We define projections from $\mathbf{J}_\infty$ to $\mathbb{R}^n$ by

$$x(\mathbf{i}) = \lim_{k \to \infty} S_{\mathbf{i}|_k}(y) \equiv \lim_{k \to \infty} S_{i_1} \circ \ldots \circ S_{i_k}(y); \tag{2.2}$$

since the S_i are contractions this limit exists and is independent of $y \in \mathbb{R}^n$. (Note that $\mathbf{i} \mapsto x(\mathbf{i})$ is not necessarily an injection.) The attractor satisfying (1.1) is then given by

$$F = \bigcup_{\mathbf{i} \in \mathbf{J}_\infty} x(\mathbf{i}). \tag{2.3}$$

A tree $\mathbf{T}$ identifies a compact subset of F. Thus we write

$$F(\mathbf{T}) = \bigcup_{\mathbf{i} \in \mathbf{T}_\infty} x(\mathbf{i}). \tag{2.4}$$

From the constructive point of view, if $B \subset \mathbb{R}^n$ is a closed ball sufficiently large to ensure that $S_i(B) \subseteq B$ for all i, then $F(\mathbf{T})$ may be obtained by the usual iterative method of producing IFS attractors, with the proviso that iterated images $S_{\mathbf{i}}(B)$ corresponding to $\mathbf{i} \notin \mathbf{T}$ are omitted. Thus $F(\mathbf{T})$ is the intersection of a decreasing sequence of closed sets

$$F(\mathbf{T}) = \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{i}\in\mathbf{T}_k} S_{\mathbf{i}}(B).$$

Here we will be concerned with self-affine sets and their subsets. Fix non-singular linear contractions $T_1, \ldots, T_N : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{a} \equiv (a_1, \ldots, a_N) \in \mathbb{R}^{nN}$ be a set of translation vectors and let $S_i : \mathbb{R}^n \to \mathbb{R}^n$ be the affine contractions

$$S_i(x) = T_i(x) + a_i, \qquad i = 1, \ldots, N. \tag{2.5}$$

We write $F(\mathbf{a}) \equiv F$ for the self-affine attractor satisfying (1.1) to emphasise its dependence on $\mathbf{a}$. The projection maps depend on $\mathbf{a}$, so writing $x_{\mathbf{a}}(\mathbf{i}) = \lim_{k\to\infty} S_{\mathbf{i}|_k}(0)$ we have

$$F(\mathbf{a}) = \bigcup_{\mathbf{i}\in\mathbf{J}_\infty} x_{\mathbf{a}}(\mathbf{i}). \tag{2.6}$$

As before, trees specify subsets of $F(\mathbf{a})$ and for a tree $\mathbf{T}$ we write

$$F(\mathbf{a}, \mathbf{T}) = \bigcup_{\mathbf{i}\in\mathbf{T}_\infty} x_{\mathbf{a}}(\mathbf{i}). \tag{2.7}$$

Our ultimate aim is to find the dimensions of $F(\mathbf{a}, \mathbf{T})$ where $\mathbf{T}$ is a random tree.

We recall the basic results on the dimensions of self-affine sets. The *singular values* $\alpha_i \equiv \alpha_i(T)$ $(i = 1, \ldots, n)$ of a linear contraction $T : \mathbb{R}^n \to \mathbb{R}^n$ are the positive square roots of the eigenvalues of TT^*, where T^* is the adjoint of T. Equivalently they are the lengths of the principal semi-axes of the image $T(B)$ of the unit ball B. With the convention that $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$ the *singular value function* $\phi^s(T)$ is defined for $0 \leq s \leq n$ as

$$\phi^s(T) = \alpha_1\alpha_2\cdots\alpha_{m-1}\alpha_m^{s-m+1},$$

where m is the integer such that $m - 1 < s \leq m$; for convenience we set $\phi^s(T) = (\alpha_1\alpha_2\cdots\alpha_n)^{s/n}$ if $s \geq n$. Under various conditions the unique number s satisfying

$$\lim_{k\to\infty} \left[\sum_{\mathbf{i}\in\mathbf{J}_k} \phi^s(T_{\mathbf{i}})\right]^{1/k} = 1. \tag{2.8}$$

gives the Hausdorff and box dimension of the self-affine set of the IFS with S_i given by (2.5). In particular, if the S_i have $\|T_i\| < \frac{1}{2}$ for all i then (2.8) gives the dimensions of $F(\mathbf{a})$ for almost all $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{R}^{nN}$ in the sense of nN-dimensional Lebesgue measure, see [5, 6, 13, 18].

We will be concerned with the random subset $F(\mathbf{a}, \mathbf{T})$ of $F(\mathbf{a})$ defined in terms of the random subtree $\mathbf{T}$ of $\mathbf{J}$ constructed as follows. Let $p_i \in (0, 1), i = 1, \ldots, N$ be probabilities. We construct $\mathbf{T}$ by defining $\mathbf{T}_k$ inductively. Let $\mathbf{T}_0$ be the null sequence $\emptyset$. Suppose that the kth-level vertices $\mathbf{T}_k$ have been selected. The $(k+1)$-level vertices $\mathbf{T}_{k+1}$ are obtained as follows. If $\mathbf{i} \in \mathbf{J}_k \setminus \mathbf{T}_k$ then $\mathbf{i}i \notin \mathbf{T}_{k+1}$. If $\mathbf{i} \in \mathbf{T}_k$ then $\mathbf{i}i \in \mathbf{T}_{k+1}$ are selected with probability p_i (not necessarily independently) for $i = 1, \ldots, N$. We assume that these selections are independent for $\mathbf{i} \neq \mathbf{j}$ (but not necessarily identically distributed). Continuing in this way, we obtain the random tree $\mathbf{T} = \bigcup_{k=0}^{\infty} \mathbf{T}_k$ with corresponding random infinite tree $\mathbf{T}_\infty$.

More formally, let Ω be the set of all subtrees of $\mathbf{J}$. For $k = 1, 2, \ldots$,let $\mathscr{F}_k$ be a sequence of σ-algebras of subsets of Ω as follows. Define an equivalence relation $\sim_k$ on Ω by $\mathbf{T} \sim_k \mathbf{T}'$, if $\mathbf{T}_m = \mathbf{T}'_m$ for $m = 0, 1, \ldots, k$, where $\mathbf{T}, \mathbf{T}' \in \Omega$ (so two trees are equivalent if they are identical down to the kth-level vertices). Let $\mathscr{F}_k$ be the (finite) set of finite unions of equivalence classes under $\sim_k$. Thus $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots$ and we let $\mathscr{F}$ be the $\sigma-$algebra generated by $\cup_{k=0}^{\infty} \mathscr{F}_k$. We assume that the probability measure P on $(\Omega, \mathscr{F})$ which defines the random trees satisfies the condition: for all $\mathbf{i} \in \mathbf{J}$ and $i \in \{1, \ldots, N\}$

$$\mathsf{P}(\mathbf{i}i \in \mathbf{T} | \mathscr{F}_k) = \begin{cases} p_i & \text{if } \mathbf{i} \in \mathbf{T} \\ 0 & \text{if } \mathbf{i} \notin \mathbf{T}. \end{cases} \tag{2.9}$$

Thus, if $\mathbf{i} \notin \mathbf{T}$, then $\mathbf{i}i \notin \mathbf{T}$ for all $i \in \{1, \ldots, N\}$ almost surely. It is easy to see that these are many possibilities for P that satisfy this condition.

In Section 3 we show how to construct a random measure on the random tree $\mathbf{T}$ which will enable us to use the potential theoretic method to find the Hausdorff and box-counting dimensions of the random set $F(\mathbf{a}, \mathbf{T})$ defined in terms of this random tree.

3 A random measure on the random tree

In this section we construct a random measure on $\mathbf{J}_\infty$ that is supported by the random subset $\mathbf{T}_\infty$. The measure of the cylinders $\mathcal{C}_\mathbf{i}$ will be subbordinate to the singular value functions of the $T_\mathbf{i}$ for $\mathbf{i} \in \mathbf{T}$. We write $T_\mathbf{i} = T_{i_1} \ldots T_{i_k}$ and $p_\mathbf{i} = p_{i_1} \ldots p_{i_k}$ for $\mathbf{i} = (i_1 \ldots i_k) \in \mathbf{J}$ in the usual way. Expectation will be denoted by E.

Lemma 3.1. *With notation as above*

$$\mathsf{E}\left(\sum_{\mathbf{i} \in \mathbf{T}_k} \phi^s(T_\mathbf{i}) \right) = \sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_\mathbf{i}) p_\mathbf{i}. \tag{3.10}$$

Proof. This is immediate, as each $\mathbf{i} \in \mathbf{J}_k$ has probability $p_\mathbf{i}$ of being in $\mathbf{T}_k$. $\square$

The critical exponent d, which is defined by (3.11) in the following lemma, is central in the analysis that follows and turns out to be the almost sure dimension of our random subsets.

Lemma 3.2. *The limit* $\lim_{k\to\infty} [\mathsf{E}(\sum_{\mathbf{i} \in \mathbf{T}_k} \phi^s(T_\mathbf{i}))]^{1/k}$ *exists and is strictly decreasing in s for $s \geq 0$. Furthermore, if $\sum_{i=1}^{n} p_i \geq 1$, there is a unique* *number $d \geq 0$ such that*

$$\lim_{k\to\infty}\Big[\mathsf{E}\Big(\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^d(T_{\mathbf{i}})\Big)\Big]^{1/k} = \lim_{k\to\infty}\Big[\sum_{\mathbf{i}\in\mathbf{J}_k}\phi^d(T_{\mathbf{i}})p_{\mathbf{i}}\Big]^{1/k} = 1. \tag{3.11}$$

Proof. The left hand equation holds by Lemma 3.1. By the submultiplicity of $\sum_{\mathbf{i}\in\mathbf{J}_k}\phi^s(T_{\mathbf{i}})$ and the continuity and strict monotonicity of $\lim_{k\to\infty}[\sum_{\mathbf{i}\in\mathbf{J}_k}\phi^s(T_{\mathbf{i}})]^{1/k}$, see [5, Proposition 4.1], there exists an unique $d \geq 0$ such that

$$\lim_{k\to\infty}\Big[\sum_{\mathbf{i}\in\mathbf{J}_k}\phi^d(T_{\mathbf{i}})p_{\mathbf{i}}\Big]^{1/k} = 1.$$

□

We assume henceforth that $\sum_{i=1}^N p_i \geq 1$, otherwise d is not defined and $\mathbf{T}$ becoming extinct (i.e. $\mathbf{T}_k = \emptyset$ for large k) with probability 1, a case that is obviously not of interest.

Lemma 3.3. *If $s > d$, then, almost surely, $\sum_{k=0}^\infty \sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_{\mathbf{i}})$ is finite with $\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_{\mathbf{i}}) \to 0$ as $k \to \infty$,.*

Proof. Let $X_k^s = \sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_{\mathbf{i}})$. For $s > d$,

$$\lim_{k\to\infty}\Big[\mathsf{E}(X_k^s)\Big]^{1/k} = \lim_{k\to\infty}\Big[\mathsf{E}\Big(\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_{\mathbf{i}})\Big)\Big]^{1/k} < 1,$$

so for some γ with $0 < \gamma < 1$ and $c > 0$, we have that $\mathsf{E}(X_k^s) \leq c\gamma^k$ for all $k \in \mathbb{N}$. Thus $\mathsf{E}\big(\sum_{k=1}^\infty (X_k^s)\big) = \sum_{k=1}^\infty \mathsf{E}(X_k^s) \leq c\sum_{k=1}^\infty \gamma^k < \infty$ and the conclusion follows. □

We first define deterministic measures $\mathcal{M}^s$ on $\mathbf{J}_\infty$ in terms of coverings of $\mathbf{T}$ by cylinders using Carathéodory's construction. For each s, define

$$\mathcal{M}_k^s(A) = \inf\Big\{\sum_{\mathbf{i}} p_{\mathbf{i}}\phi^s(T_{\mathbf{i}}) : A \subseteq \bigcup \mathcal{C}_{\mathbf{i}}, |\mathbf{i}| \geq k\Big\}$$

for $k \geq 0$ and $A \subseteq \mathbf{J}_\infty$, and

$$\mathcal{M}^s(A) = \lim_{k\to\infty}\mathcal{M}_k^s(A).$$

Then $\mathcal{M}^s$ is a regular Borel measure obtained by Carathéodory's construction, see [17]. The following lemma states that d is also the critical parameter for $\mathcal{M}^s(\mathbf{J}_\infty)$.

Lemma 3.4. *If $0 < s < d$, then $\mathcal{M}^s(\mathbf{J}_\infty) > 0$.*

Proof. The proof, which depends on the submultiplicativity of $p_{\mathbf{i}}\phi^s(T_{\mathbf{i}})$ is virtually identical to that of [5, Proposition 4.1], except that here we work with $p_{\mathbf{i}}\phi^s(T_{\mathbf{i}})$ rather than $\phi^s(T_{\mathbf{i}})$. □

We fix $0 < t < d$ for the rest of the section. We now replace $\mathcal{M}^t$ by a subordinate measure μ that is positive and finite.

Proposition 3.5. *Let $0 < t < d$. Then $\mathcal{M}^t(\mathbf{J}_\infty) > 0$ and there exists a Borel measure μ on $\mathbf{J}_\infty$ such that $0 < \mu(\mathbf{J}_\infty) < \infty$ and*

$$\mu(\mathcal{C}_{\mathbf{i}}) \leq p_{\mathbf{i}} \phi^t(T_{\mathbf{i}}) \qquad (\mathbf{i} \in \mathbf{J}). \tag{3.12}$$

Proof. The proof follows the standard method of reducing positive measures to positive finite measures and may be proved in a similar way to [3, Theorem 5.4] or deduced from [17, Theorem 54].Full details of the proof in this context may be found in [16, Proposition1.11]. □

We use a martingale argument to obtain random measures μ_∞ that are supported by the random trees $\mathbf{T}_\infty$.

For $k \in \mathbb{N}$ we define random measures μ_k on the cylinders $\{\mathcal{C}_{\mathbf{q}} : |\mathbf{q}| \leq k\}$ by setting

$$\mu_k(\mathcal{C}_{\mathbf{q}}) = \sum_{\mathbf{q} \preceq \mathbf{i} \in \mathbf{T}_k} \frac{\mu(\mathcal{C}_{\mathbf{i}})}{p_{\mathbf{i}}}, \tag{3.13}$$

so if $\mathbf{q} \notin \mathbf{T}$ then $\mu_k(\mathcal{C}_{\mathbf{q}}) = 0$. Clearly μ_k is $\mathscr{F}_k$ measurable and additive on $\{\mathcal{C}_{\mathbf{q}} : |\mathbf{q}| \leq k\}$.

Lemma 3.6. *Write $q = |\mathbf{q}|$ for $\mathbf{q} \in \mathbf{J}$. For all $\mathbf{q} \in \mathbf{J}$, $\{\mu_k(\mathcal{C}_{\mathbf{q}})\}_{k=q}^{\infty}$ is an L^2 martingale with respect to $\mathscr{F}_k$. Thus we may define $\mu_\infty(\mathcal{C}_{\mathbf{q}}) = \lim_{k\to\infty} \mu_k(\mathcal{C}_{\mathbf{q}})$, the limit existing almost surely for all $\mathbf{q} \in \mathbf{J}$. Then μ_∞ is a random measure on the Borel subsets of $\mathbf{J}_\infty$ that is supported by $\mathbf{T}_\infty$. In particular, for all $\mathbf{q} \in \mathbf{J}$,*

$$\mathsf{E}\big(\mu_\infty(\mathcal{C}_{\mathbf{q}})\big) = \mu(\mathcal{C}_{\mathbf{q}}),$$

and

$$\mathsf{E}\big(\mu_\infty(\mathcal{C}_{\mathbf{q}})^2\big) \leq c\phi^t(T_{\mathbf{q}})\mu(\mathcal{C}_{\mathbf{q}}), \tag{3.14}$$

where c is a constant independent of $\mathbf{q}$, and $0 < \mu_\infty(\mathbf{J}_\infty) < \infty$ with positive probability.

Proof. For each $\mathbf{q} \in \mathbf{T}$,

$$\begin{aligned}
\mathsf{E}(\mu_{k+1}(\mathcal{C}_{\mathbf{q}})|\mathscr{F}_k) &= \mathsf{E}\bigg(\sum_{\mathbf{q} \preceq \mathbf{i}i \in \mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_{\mathbf{i}i}} \bigg| \mathscr{F}_k \bigg) && \text{by (3.13)} \\
&= \sum_{\mathbf{q} \preceq \mathbf{i} \in \mathbf{T}_k} p_{\mathbf{i}}^{-1} \mathsf{E}\bigg(\sum_{i:\mathbf{i}i \in \mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_i} \bigg) \\
&= \sum_{\mathbf{q} \preceq \mathbf{i} \in \mathbf{T}_k} p_{\mathbf{i}}^{-1} \sum_{i=1}^{N} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_i} \cdot p_i && \text{by (2.9)} \\
&= \sum_{\mathbf{q} \preceq \mathbf{i} \in \mathbf{T}_k} \mu(\mathcal{C}_{\mathbf{i}})/p_{\mathbf{i}} \\
&= \mu_k(\mathcal{C}_{\mathbf{q}}),
\end{aligned}$$

so $(\mu_k(\mathcal{C}_{\mathbf{q}}), \mathscr{F}_k)_{k\geq q}$ is a non-negative martingale.

By the Martingale Convergence Theorem [19], for each $\mathbf{q}$, there exists a random variable $\mu_\infty(\mathcal{C}_{\mathbf{q}}) \geq 0$ such that, $\mu_k(\mathcal{C}_{\mathbf{q}}) \to \mu_\infty(\mathcal{C}_{\mathbf{q}})$ almost surely.

Then μ_k is additive on any finite set of cylinders for k sufficiently large, so μ_∞ is almost surely finitely additive on the cylinders and extends to a measure on the Borel subsets of $\mathbf{T}_\infty$.

To show that $\mu_\infty(\mathbf{T}_\infty) > 0$ with positive probability, it is enough to show that $\mu_k(\mathcal{C}_{\mathbf{q}})$ is L^2 bounded, i.e. $\sup_k \mathsf{E}(\mu_k(\mathcal{C}_{\mathbf{q}})^2) < \infty$. We show more generally that

$$\mathsf{E}\left(\mu_k(\mathcal{C}_{\mathbf{q}})^2\right) \leq c\phi^t(T_{\mathbf{q}})\mu(\mathcal{C}_{\mathbf{q}}) \quad (k \geq q),$$

where c does not depend on k or $\mathbf{q}$.

For $k \geq q$, taking conditional expectations of $\mu_{k+1}(\mathcal{C}_{\mathbf{q}})^2$,

$$\begin{aligned}
\mathsf{E}\left(\mu_{k+1}(\mathcal{C}_{\mathbf{q}})^2 \middle| \mathscr{F}_k\right) &= \mathsf{E}\left(\Big(\sum_{\substack{\mathbf{i}i\in\mathbf{T}_{k+1}\\ \mathbf{q}\preceq\mathbf{i}}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_{\mathbf{i}i}}\Big)^2 \middle| \mathscr{F}_k\right) \qquad \text{by (3.13)}\\
&= \mathsf{E}\left(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} p_{\mathbf{i}}^{-2} \sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \sum_{j:\mathbf{i}j\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})\mu(\mathcal{C}_{\mathbf{i}j})}{p_i\cdot p_j} \middle| \mathscr{F}_k\right)\\
&\quad + \mathsf{E}\left(\sum_{\substack{\mathbf{i}\neq\mathbf{j}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i},\mathbf{j}}} p_{\mathbf{i}}^{-1}p_{\mathbf{j}}^{-1} \sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \sum_{j:\mathbf{j}j\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})\mu(\mathcal{C}_{\mathbf{j}j})}{p_i\cdot p_j} \middle| \mathscr{F}_k\right)\\
&\leq \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} p_{\mathbf{i}}^{-2}\left(\sum_{1\leq i,j\leq N} \frac{\mu(\mathcal{C}_{\mathbf{i}})\mu(\mathcal{C}_{\mathbf{i}})}{p_i\cdot p_j}\right)\\
&\quad + \sum_{\substack{\mathbf{i}\neq\mathbf{j}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i},\mathbf{j}}} p_{\mathbf{i}}^{-1}p_{\mathbf{j}}^{-1} \mathsf{E}\left(\sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \sum_{j:\mathbf{j}j\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})\mu(\mathcal{C}_{\mathbf{j}j})}{p_i\cdot p_j}\right)\\
&\leq c_1 \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \left(\mu(\mathcal{C}_{\mathbf{i}})/p_{\mathbf{i}}\right)^2 + \sum_{\substack{\mathbf{i}\neq\mathbf{j}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i},\mathbf{j}}} \mu(\mathcal{C}_{\mathbf{i}})\mu(\mathcal{C}_{\mathbf{j}})p_{\mathbf{i}}^{-1}p_{\mathbf{j}}^{-1} \qquad \text{by (2.9) and independence}\\
&\leq c_1 \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \left(\mu(\mathcal{C}_{\mathbf{i}})/p_{\mathbf{i}}\right)^2 + \left(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \mu(\mathcal{C}_{\mathbf{i}})/p_{\mathbf{i}}\right)^2\\
&\leq c_1 \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \left(\mu(\mathcal{C}_{\mathbf{i}})/p_{\mathbf{i}}\right)^2 + \mu_k(\mathcal{C}_{\mathbf{q}})^2, \qquad \text{by (3.13)}
\end{aligned}$$

where $c_1 = \sum_{i,j=1}^N p_i^{-1}p_j^{-1}$. Taking conditional expectations again, it follows that, for $k \geq q$,

$$\begin{aligned}
\mathsf{E}\left(\mu_{k+1}(\mathcal{C}_{\mathbf{q}})^2 \middle| \mathscr{F}_q\right) &= \mathsf{E}\left(\mathsf{E}\left(\mu_{k+1}(\mathcal{C}_{\mathbf{q}})^2 \middle| \mathscr{F}_k\right) \middle| \mathscr{F}_q\right)\\
&\leq c_1 \mathsf{E}\left(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \left(\mu(\mathcal{C}_{\mathbf{i}})/p_{\mathbf{i}}\right)^2 \middle| \mathscr{F}_q\right) + \mathsf{E}\left(\mu_k(\mathcal{C}_{\mathbf{q}})^2 \middle| \mathscr{F}_q\right) \qquad (3.15)
\end{aligned}$$

Consider the first term of this sum for $k \geq q$. Set $\alpha_+ = \max_{i=1,\dots,N}\{\alpha_1(T_i)\} < 1$. If $\mathbf{q} \notin \mathbf{T}_q$ then $\mathsf{E}\big(\sum_{\mathbf{i}\in\mathbf{T}_k,\mathbf{q}\preceq\mathbf{i}} \mu(\mathcal{C}_\mathbf{i})^2 \big| \mathscr{F}_q\big) = 0$. For $\mathbf{q} \in \mathbf{T}_q$, we have

$$\begin{aligned}
\mathsf{E}\Big(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \big(\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i}\big)^2 \big| \mathscr{F}_q\Big) &= \sum_{\substack{\mathbf{i}\in\mathbf{J}_k\\ \mathbf{q}\preceq\mathbf{i}}} \big(\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i}\big)^2 p_{i_{q+1}} \dots p_{i_k} \\
&\leq \sum_{\substack{\mathbf{i}\in\mathbf{J}_k\\ \mathbf{q}\preceq\mathbf{i}}} \mu(\mathcal{C}_\mathbf{i}) p_\mathbf{i}^{-1} \cdot \phi^t(T_\mathbf{i})\ p_\mathbf{i}\ p_\mathbf{q}^{-1} \qquad \text{by (3.12)} \\
&\leq p_\mathbf{q}^{-1} \sum_{\mathbf{i}'\in\mathbf{J}_{k-q}} \mu(\mathcal{C}_{\mathbf{q}\mathbf{i}'})\phi^t(T_{\mathbf{q}\mathbf{i}'}) \qquad (3.16) \\
&\leq p_\mathbf{q}^{-1} \sum_{\mathbf{i}'\in\mathbf{J}_{k-q}} \mu(\mathcal{C}_{\mathbf{q}\mathbf{i}'})\phi^t(T_\mathbf{q})\phi^t(T_{\mathbf{i}'}) \\
&\leq p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}) \sum_{\mathbf{i}'\in\mathbf{J}_{k-q}} \frac{\mu(\mathcal{C}_{\mathbf{q}\mathbf{i}'})}{\mu(\mathcal{C}_\mathbf{q})}\phi^t(T_{\mathbf{i}'}) \\
&\leq p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})\alpha_+^{t(k-q)}, \qquad (3.17)
\end{aligned}$$

provided that $\mu(\mathcal{C}_\mathbf{q}) \neq 0$, otherwise (3.17) follows from (3.16). Hence iterating (3.15) and using (3.17), with $\mathbf{q} \in \mathbf{T}_k$, we obtain

$$\begin{aligned}
\mathsf{E}\Big(\mu_{k+1}(\mathcal{C}_\mathbf{q})^2 \Big| \mathscr{F}_q\Big) &\leq c_1 p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})\alpha_+^{t(k-q)} + \mathsf{E}\big(\mu_k(\mathcal{C}_\mathbf{q})^2 \big| \mathscr{F}_q\big) \\
&\leq c_1 p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})\sum_{l=q}^{k}\alpha_+^{t(l-q)} + \mathsf{E}\big(\mu_q(\mathcal{C}_\mathbf{q})^2 \big| \mathscr{F}_q\big) \\
&\leq c_2 p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}) + \mathsf{E}\big(\mu_q(\mathcal{C}_\mathbf{q})^2 \big| \mathscr{F}_q\big), \qquad (3.18)
\end{aligned}$$

where $c_2 = c_1 \sum_{l=q}^{\infty} \alpha_+^{t(l-q)} < \infty$. By definition (3.13) and inequality (3.12)

$$\begin{aligned}
\mathsf{E}\Big(\mu_q(\mathcal{C}_\mathbf{q})^2 \Big| \mathscr{F}_q\Big) &= \mu_q(\mathcal{C}_\mathbf{q})^2 = p_\mathbf{q}^{-2}\mu(\mathcal{C}_\mathbf{q})^2 \\
&\leq p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}). \qquad (3.19)
\end{aligned}$$

Thus by (3.18), we have that, for $k \geq q$,

$$\mathsf{E}\Big(\mu_k(\mathcal{C}_\mathbf{q})^2 \Big| \mathscr{F}_q\Big) \leq c p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}),$$

where $c = (c_2 + 1)$. Finally, we get the unconditional expectation

$$\begin{aligned}
\mathsf{E}\left(\mu_k(\mathcal{C}_\mathbf{q})^2\right) &= p_\mathbf{q}\mathsf{E}\left(\mu_k(\mathcal{C}_\mathbf{q})^2 \big| \mathscr{F}_q\right) \\
&\leq c\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}).
\end{aligned}$$

In particular, $\mu_k(\mathcal{C}_\emptyset)$ is L^2 bounded, so the standard property of L^2 martingales, see [19], it follows that $\mu_\infty(\mathbf{J}_\infty) > 0$ with positive probability, and $\mathsf{E}\left(\mu_\infty(\mathcal{C}_\emptyset)\right) = \mathsf{E}\left(\mu_k(\mathcal{C}_\emptyset)\right) = \mu(\mathcal{C}_\emptyset) > 0$.

Since $\mu_k(\mathcal{C}_\mathbf{q}) \to \mu_\infty(\mathcal{C}_\mathbf{q})$ almost surely for all $\mathbf{q}$, by Fatou's Lemma,

$$\mathsf{E}\left(\mu_\infty(\mathcal{C}_\mathbf{q})^2\right) \leq \varliminf_{k\to\infty} \mathsf{E}\left(\mu_k(\mathcal{C}_\mathbf{q})^2\right) \leq c\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}). \qquad (3.20)$$

□

4 The dimensions of random subsets of self-affine fractals

We use the results of Section 3 to find almost sure dimensions of random subsets of self-affine attractors. Keeping the notation of Section 2, $S_i(x) = T_i(x) + a_i$ are affine maps with linear parts T_i and translations a_i and $x_{\mathbf{a}}(\mathbf{i}) = \lim_{k\to\infty} S_{\mathbf{i}|_k}(0)$ are the projections from $\mathbf{J}_\infty$ to $\mathbb{R}^n$. For $\mathbf{T}$ a tree with boundary $\mathbf{T}_\infty$ recall (2.7) that

$$F(\mathbf{a}, T) = \bigcup_{\mathbf{i}\in\mathbf{T}_\infty} x_{\mathbf{a}}(\mathbf{i}). \tag{4.21}$$

We take $\mathbf{T}$ to be a random tree constructed as in (2.9) so that for each $\mathbf{a}$, $F(\mathbf{a}, T)$ is a random subset of the deterministic self-affine attractor $F(\mathbf{a})$ of the IFS $\{S_1, \ldots, S_N\}$.

We investigate the Hausdorff and box-counting dimensions of $F(\mathbf{a}, T)$. The proof of the upper bound is a random variation of that for the deterministic self-affine set $F(\mathbf{a})$ in [5, Theorem 5.4]. As usual, $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{B}}$ denote Hausdorff and box-counting dimensions, with $\underline{\dim}_{\mathrm{B}}$ and $\overline{\dim}_{\mathrm{B}}$ denoting lower and upper box-counting dimensions.

Theorem 4.1. *Let* $\mathbf{T}$ *be the random tree as in* (2.9) *and let* $F(\mathbf{a}, T)$ *be the random subset* (4.21) *of the self-affine set* $F(\mathbf{a})$*. Then, almost surely,* $\dim_{\mathrm{H}} F(\mathbf{a}, T) \le \underline{\dim}_{\mathrm{B}} F(\mathbf{a}, T) \le \overline{\dim}_{\mathrm{B}} F(\mathbf{a}, T) \le \min\{d, n\}$ *for all* $\mathbf{a} \in \mathbb{R}^{nN}$*, where* d *is given by* (3.11).

Proof. The result is clear if $d \ge n$. Let s be such that $d < s < n$. By Lemma 3.3 there exists almost surely a finite random constant C_1 such that

$$\sum_{k=1}^{\infty} \sum_{\mathbf{i}\in\mathbf{T}_k} \phi^s(T_{\mathbf{i}}) \le C_1. \tag{4.22}$$

Let B be a ball large enough to ensure that $S_i(B) \subset B$ for $i = 1, \ldots, N$. Let m be the least integer greater than or equal to s. Given $0 < r < \sqrt{n}|B|$, for every $\mathbf{i} \in \mathbf{T}_\infty$ let $\mathbf{i}' = i_1 \ldots i_q$ be the finite sequence obtained by curtailing $\mathbf{i}$ after q terms, where q is the smallest positive integer such that $r \ge \sqrt{n}|B|\alpha_m(T_{i_1\ldots i_q}) > r\alpha_-^n$ with $\alpha_m(T_{i_1\ldots i_q})$ the mth singular value of $T_{\mathbf{i}'}$ and $\alpha_- = \min_{1\le i\le N}\{\alpha_n(T_i)\}$. Letting $A = \{\mathbf{i}' \in \mathbf{T} : \mathbf{i} \in \mathbf{T}_\infty\}$, we have, for all $\mathbf{i} \in \mathbf{T}_\infty$, that $\mathbf{i}|_k \in A$ for some k, so $F(\mathbf{a}, \mathbf{T}) \subseteq \bigcup_{\mathbf{i}'\in A} S_{\mathbf{i}'}(B)$. Moreover by (4.22), we have that, almost surely, $\sum_{\mathbf{i}'\in A} \phi^s(T_{\mathbf{i}'}) \le C_1 < \infty$.

Just as in [5, Theorem 5.4] we note that each ellipsoid $S_{\mathbf{i}'}(B)$ for $\mathbf{i}' \in A$ may be covered by at most $c_1\phi^s(T_{\mathbf{i}'})\big(\alpha_m(T_{\mathbf{i}'})\big)^{-s} \le c_2\phi^s(T_{\mathbf{i}'})r^{-s}$ sets of diameter r, where c_1, c_2 are independent of $\mathbf{i}'$ and r. The aggregate of these covers provides a cover of $F(\mathbf{a}, \mathbf{T})$ by $c_2 \sum_{\mathbf{i}'\in A} \phi^s(T_{\mathbf{i}'}) \le c_2 C_1 r^{-s}$ sets of diameter r. This implies that $\overline{\dim}_{\mathrm{B}} F(\mathbf{a}, T) \le s$, and the conclusion follows since this is true for all $d < s < n$. □

Proposition 4.2. *Assume that* $\max_{1\le i\le N}\|T_i\| < \frac{1}{2}$. *Let* $F(\mathbf{a},\mathbf{T})$ *be as above. For all* $t < \min\{d,n\}$, *conditional on* $\mu_\infty(\mathbf{T}_\infty) > 0$, *almost surely* $\dim_{\mathrm{H}} F(\mathbf{a}) \ge t$ *for almost all* $\mathbf{a} \in \mathbb{R}^{nN}$. *(Note that* μ_∞, *defined by Lemma 3.6, depends on* t *and for each such* t *there is a positive probability that* $\mu_\infty(\mathbf{T}_\infty) > 0$.*)*

Proof. This is an adaptation of the proof of [5][Theorem5.3] to the random setting. Let $0 < t < \min\{n,d\}$. Let μ be as in (3.12) with $0 < \mu(\mathbf{J}_\infty) < \infty$. By Lemma 3.6 there exists a random measure μ_∞ supported by $\mathbf{T}_\infty$ such that, with positive probability, $0 < \mu_\infty(\mathbf{J}_\infty) < \infty$ and such that $\mathsf{E}\left(\mu_\infty(\mathcal{C}_\mathbf{q})^2\right) \le c\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})$for all $\mathbf{q}$.

Let $0 < s < t$ with s not an integer. It was shown in [5, 18] that, for such s and $\|T_i\| < \frac{1}{2}$ for all $1 \le i \le N$, there is a number c_1 such that

$$\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mathbf{a}}{|x_\mathbf{a}(\mathbf{i}) - x_\mathbf{a}(\mathbf{j})|^s} \le \frac{c_1}{\phi^s(T_{\mathbf{i}\wedge\mathbf{j}})}$$

for all distinct $\mathbf{i}, \mathbf{j} \in \mathbf{J}_\infty$. Writing $\mathbf{q} = \mathbf{i}\wedge\mathbf{j}$,

$$\begin{aligned}
\mathsf{E}\left(\int_{\mathbf{J}_\infty}\int_{\mathbf{J}_\infty}\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mu_\infty(\mathbf{i})d\mu_\infty(\mathbf{j})d\mathbf{a}}{|x_\mathbf{a}(\mathbf{i}) - x_\mathbf{a}(\mathbf{j})|^s}\right) &\le c_1\mathsf{E}\left(\int_{\mathbf{J}_\infty}\int_{\mathbf{J}_\infty} \frac{d\mu_\infty(\mathbf{i})d\mu_\infty(\mathbf{j})}{\phi^s(T_{\mathbf{i}\wedge\mathbf{j}})}\right)\\
&\le c_1\mathsf{E}\left(\sum_{\mathbf{q}\in\mathbf{J}}\sum_{i\ne j}\frac{\mu_\infty(\mathcal{C}_{\mathbf{q}i})\mu_\infty(\mathcal{C}_{\mathbf{q}j})}{\phi^s(T_\mathbf{q})}\right)\\
&\le c_1\mathsf{E}\left(\sum_{\mathbf{q}\in\mathbf{J}}\frac{\mu_\infty(\mathcal{C}_\mathbf{q})^2}{\phi^s(T_\mathbf{q})}\right)\\
&\le c_1 c\sum_{k=0}^{\infty}\sum_{\mathbf{q}\in\mathbf{J}_k}\frac{\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})}{\phi^s(T_\mathbf{q})}\\
&\le c_1 c\sum_{k=0}^{\infty}\sum_{\mathbf{q}\in\mathbf{J}_k}\alpha_+^{k(t-s)}\mu(\mathcal{C}_\mathbf{q})\\
&\le c_1 c\mu(\mathbf{J}_\infty)\sum_{k=0}^{\infty}\alpha_+^{k(t-s)}\\
&< \infty
\end{aligned}$$

where $\alpha_+ = \max_{1\le i\le N}\{\alpha_1(T_i)\}$ and we have used Lemma 3.6. It follows that, with probability 1,

$$\int_{\mathbf{J}_\infty}\int_{\mathbf{J}_\infty}\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mu_\infty(\mathbf{i})d\mu_\infty(\mathbf{j})d\mathbf{a}}{|x_\mathbf{a}(\mathbf{i}) - x_\mathbf{a}(\mathbf{j})|^s} < \infty. \tag{4.23}$$

Applying Fubini's theorem,

$$\int_{\mathbf{J}_\infty}\int_{\mathbf{J}_\infty} \frac{d\mu_\infty(\mathbf{i})d\mu_\infty(\mathbf{j})}{|x_\mathbf{a}(\mathbf{i}) - x_\mathbf{a}(\mathbf{j})|^s} < \infty$$

for almost all $\mathbf{a} \in B(0,\rho)$. Given $\mathbf{a} \in B(0,\rho)$, for each realization of μ_∞, we define a measure on $\mathbb{R}^n$ by

$$\nu(A) = \mu_\infty\{\mathbf{i} \in \mathbf{J}_\infty : x_{\mathbf{a}}(\mathbf{i}) \in A\}, \qquad (A \subseteq \mathbb{R}^n).$$

By the continuity of $\mathbf{i} \mapsto x_{\mathbf{a}}(\mathbf{i})$, ν is a random Borel measure on $\mathbb{R}^n$ supported by the random set $F(\mathbf{a},\mathbf{T})$ with $\nu(\mathbb{R}^n) = \mu_\infty(\mathbf{J}_\infty)$. Conditional on $\mu_\infty(\mathbf{T}_\infty) > 0$ the potential-theoretic characterization of Hausdorff dimension, see [7], gives that $\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) \geq s$ for almost all $\mathbf{a}$. □

We now assume more regularity for the random tree construction than just (2.9), namely that the probability distribution of $\{i : \mathbf{i}i \in \mathbf{T}\}$ is the same for all $\mathbf{i} \in \mathbf{T}$. The random sets $F(\mathbf{a},\mathbf{T})$ may then be thought of as *statistically self affine*. In particular,

$$\mathsf{P}\big(\sharp\{i : \mathbf{i}i \in \mathbf{T}_{k+1} | \mathbf{i} \in \mathbf{T}_k\} = r\big) \equiv \mathsf{P}(\sharp\mathbf{T}_1 = r)$$

for all $\mathbf{i} \in \mathbf{T}_k$ and $r = 1,\ldots,N$, so that $\sharp\mathbf{T}_k$ is a Galton-Watson branching process, see [1]. By standard theory, the probability η of extinction, that is of $\mathbf{T}_k = 0$ for all sufficiently large k, is given by the smallest non-negative root of the generating function equation

$$\eta = \sum_{r=0}^{\infty} \mathsf{P}\big(\sharp(\mathbf{T}_1) = r\big)\eta^r.$$

Theorem 4.3. *Assume that* $\max_{1\leq i\leq N} \|T_i\| < \frac{1}{2}$. *Let* $F(\mathbf{a},\mathbf{T})$ *be as above. With probability* η, *we have* $F(\mathbf{a},\mathbf{T}) = \emptyset$ *for all* $\mathbf{a}$ *and with probability* $1-\eta > 0$ *we have* $\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) = \dim_{\mathrm{B}} F(\mathbf{a},\mathbf{T}) = \min\{d,n\}$ *for almost all* $\mathbf{a} \in \mathbb{R}^{nN}$, *where* d *is given by* (3.11).

Proof. Clearly if the branching process becomes extinct then $F(\mathbf{a},\mathbf{T}) = \emptyset$.

Otherwise, conditional on non-extinction, $\sharp\mathbf{T}_k \to \infty$ almost surely. Let $t < \min\{d,n\}$. For $\mathbf{q} \in \mathbf{T}_k$, write $F(\mathbf{a},\mathbf{T}_{\mathbf{q}}) = \bigcup\{x_{\mathbf{a}}(\mathbf{i}) : \mathbf{i} \in \mathbf{T} \text{ and } \mathbf{q} \preceq \mathbf{i}\}$. By Proposition 4.2, with positive probability, p, say, $\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) \geq t$ for almost all $\mathbf{a} \in \mathbb{R}^{nN}$, so by the statistical self-affinity of the construction

$$\mathsf{P}(\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}_{\mathbf{q}})) \geq t \text{ for almost all } \mathbf{a}) = p.$$

Given $\epsilon > 0$ there is an integer m such that $(1-p)^m < \epsilon$. Conditional on non-extinction there exists q such that $\sharp\mathbf{T}_q \geq m$. Then $F(\mathbf{a},\mathbf{T}) = \bigcup_{|\mathbf{q}|=q} F(\mathbf{a},\mathbf{T}_{\mathbf{q}})$, with $\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) = \max_{|\mathbf{q}|=q}\{\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}_{\mathbf{q}})\}$ and $\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}_{\mathbf{q}})$ independent for each such $\mathbf{q}$. Thus

$$\mathsf{P}(\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) \geq t \,|\, \text{non-extinction}) \geq (1-p)^m < \epsilon.$$

This holds for all $\epsilon > 0$ and also for all $t < \min\{d,n\}$ so

$$\mathsf{P}(\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) \geq t \,|\, \text{non-extinction}) = 1.$$

Combining this with Theorem 4.1, $\dim_{\mathrm{H}} F(a) = \dim_{\mathrm{B}} F(a) = \min\{d,n\}$, conditional on non-extinction. □

5 Examples

We illustrate Theorem 4.3 by some examples. Firstly, if each vertex of the random tree has a constant number of children the possibility of extinction is avoided.

Example 5.1. *Fix an integer* $m\,(2 \le m \le N)$. *Let* $\mathbf{T}$ *be a 'constant valence' random tree, thus each of the* $\binom{N}{m}$ *possible* m *vertex subsets of* $\{\mathbf{i}i \in \mathbf{T} : 1 \le i \le N\}$ *occurs with equal probability* $1/\binom{N}{m}$. *In particular*

$$\mathsf{P}(\mathbf{i}i \in \mathbf{T} | \mathscr{F}_k) = \begin{cases} \frac{m}{N} & \text{if } \mathbf{i} \in \mathbf{T} \\ 0 & \text{if } \mathbf{i} \notin \mathbf{T} \end{cases}. \tag{5.24}$$

By Theorem 4.3, with probability 1, $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) = \dim_{\mathrm{B}} F(\mathbf{a}, \mathbf{T}) = \min\{d, n\}$ *for almost all* $\mathbf{a}$, *where* d *satisfies*

$$\lim_{k\to\infty} [\sum_{\mathbf{i}\in\mathbf{J}_k} \phi^d(T_{\mathbf{i}})]^{1/k} = \frac{N}{m}.$$

When the linear parts T_i are given by triangular matrices the almost sure dimensions are given by a closed formula.

Example 5.2. *Suppose that the linear parts of the affine mappings* (2.5) *are simultaneously represented by upper triangular matrices* T_i *with respect to some basis, with*

$$T_i = \begin{pmatrix} t_1^i & t_{12}^i & \cdots & t_{1n}^i \\ 0 & t_2^i & \cdots & t_{2n}^i \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t_n^i \end{pmatrix},$$

where $0 < |t_r^i| < \frac{1}{2}$, $r = 1, \ldots, n$. *Conditional on non-extinction of the construction, almost surely* $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) = \dim_{\mathrm{B}} F(\mathbf{a}, \mathbf{T}) = \min\{s, n\}$ *for almost all* $\mathbf{a}$, *where* s *satisfies*

$$\max_{\substack{\{j_1,\ldots,j_{m-1}\}\\ \{j'_1,\ldots,j'_m\}\\ m-1<s\le m}} \big\{|t^1_{j_1}\cdots t^1_{j_{m-1}}|^{m-s}|t^1_{j'_1}\cdots t^1_{j'_m}|^{s-m+1}p_1 + \cdots + |t^N_{j_1}\cdots t^N_{j_{m-1}}|^{m-s}|t^N_{j'_1}\cdots t^N_{j'_m}|^{s-m+1}p_N\big\} = 1, \tag{5.25}$$

where m *is the integer such that* $m-1 < s \le m$ *and the maximum is over all sets* $\{j_1, \ldots, j_{m-1}\}$ *and* $\{j'_1, \ldots, j'_m\}$ *of* $m-1$, *respectively* m, *distinct integers from* $\{1, \ldots, n\}$.

Attractors of IFSs represented by upper triangular matrices are studied in [8] where an explicit formula is obtained for (2.8). A trivial modification to include the p_i gives (5.25) as a closed form of (3.11), which by Theorem 4.3 is the almost sure dimension.

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