Dimensions of self-affine fractals and multifractals generated by upper-triangular matrices

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Abstract

We consider calculation of the dimensions of self-affine fractals and multifractals that are the attractors of iterated function systems specified in terms of upper triangular matrices. Using methods from linear algebra we obtain explicit formulae for the dimensions that are valid in many cases.

1 Introduction

An iterated function system (IFS) is a finite family of contractions $S_1, \ldots, S_N : \mathbb{R}^n \to \mathbb{R}^n$ with $N \geq 2$. It is well-known that there is a non-empty compact set $F \subseteq \mathbb{R}^n$ such that

$$F = \bigcup_{i=1}^{N} S_i(F), \tag{1.1}$$

called the *attractor* or *invariant set* of the IFS, see, for example, [6]. The contractions $S_1, \ldots, S_N$ satisfy the *open set condition* if there exists a bounded non-empty open set D such that

$$D \supseteq \bigcup_{i=1}^{N} S_i(D) \tag{1.2}$$

with the union disjoint

If the IFS consists of affine contractions S_i: $\mathbb{R}^n \to \mathbb{R}^n$

$$S_i(x) = T_i(x) + v_i, \qquad i = 1, 2, \ldots, N, \tag{1.3}$$

where $v_i \in \mathbb{R}^n$ is a translation vector and T_i is a non-singular linear mapping, we call the attractor F *self-affine*, and if the S_i are similarity mappings we call F *self-similar*.

Formulae giving the Hausdorff and box-counting dimensions of self-similar sets satisfying the open set condition are well-known, see [6]. However, calculation of the dimensions of general self-affine sets is more awkward. An expression for the dimension involving singular-value functions of iterated products of the linear mappings is known to apply in many cases, see Section 3 and [3, 4], but since this formula depends on the limiting behaviour of sums over all possible compositions of the contractions it is unwieldy to use in practice, see [2] for an algortihm where convergence is described as 'not fast'. Here we show that if the T_i are upper-triangular matrices then the formula reduces to a simple equation for the dimension that is easily solved. (A special case of this was discussed in [7].)

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear contraction. The *singular values* $\alpha_i \equiv \alpha_i(T)$ of T $(i = 1, \ldots, n)$ are the positive square roots of the eigenvalues of TT^*, where T^* is the transpose of T. Equivalently they are the lengths of the principal semi-axes of the image $T(B)$ of the unit ball B. We adopt the convention that $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$. The *singular value function* $\phi^s(T)$ is then defined for $0 \leq s \leq n$ as

$$\phi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \alpha_m^{s-m+1},$$

where m is the integer such that $m - 1 < s \leq m$, with the convention that $\phi^s(T) = (\alpha_1 \alpha_2 \cdots \alpha_n)^{s/n}$ if $s \geq n$.

Let I_k be the set of all k-term sequences $\mathbf{i} \equiv (i_1, \ldots, i_k)$ where $i_j \in \{1, 2, \ldots, N\}$ for all j, and write $|\mathbf{i}|$ for the length k of such a sequence $\mathbf{i}$. Given an IFS (1.3), we consider the products $T_{\mathbf{i}} = T_{i_1} T_{i_2} \ldots T_{i_k}$. The positive number

$$d(T_1, \ldots, T_N) = \Big\{ s : \lim_{k \to \infty} \big[\sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}) \big]^{1/k} = 1 \Big\} \tag{1.4}$$

is well-defined with the limits existing, and the Hausdorff or box-counting dimension of the self-affine attractor F is often given by $\min\{n, d(T_1, \ldots, T_N)\}$, see Section 3 where more details and examples are given. Our aim in the next section is to show that when the T_i are upper triangular matrices, $d(T_1, \ldots, T_N)$ may be expressed in a simple form depending only on the diagonal entries of the T_i, see Theorem 3.2. This enables the dimensions to be calculated easily by solving a small number of simple explicit equations. In Section 4 we extend this to self-affine multifractals.

2 Triangular matrices and singular value functions

The main result of this section is Theorem 2.5 which expresses $\sum_{|\mathbf{i}|=k} [\phi^s(T_{\mathbf{i}})]^{1/k}$ in terms of the diagonal entries of the T_i. To estimate the singular value functions involved we first recall the relationship between singular value functions and minors of matrices, using notions from exterior algebra [12].

The m-dimensional exterior algebra Λ^m consists of formal elements $v_1 \wedge \cdots \wedge v_m$ with $v_i \in \mathbb{R}^n$ such that $v_1 \wedge \cdots \wedge v_m = 0$ if $v_i = v_j$ for some $i \neq j$, and such that interchanging two different elements reverses the sign, i.e. $v_1 \wedge \cdots v_i \cdots v_j \cdots \wedge v_m = -v_1 \wedge \cdots v_j \cdots v_i \cdots \wedge v_m$, if $i \neq j$. Then Λ^m is a vector space of dimension $\binom{n}{m}$ with basis $\{e_{j_1} \wedge \cdots \wedge e_{j_m} : 1 \leq j_1 < \cdots < j_m \leq n\}$ where $e_1, e_2, \ldots, e_n$ are a given set of orthonormal vectors in $\mathbb{R}^n$.

Then Λ^m becomes a normed space under the norm

$$\|v_1 \wedge \cdots \wedge v_m\| = |m\text{-dimensional volume of the parallelepiped spanned by } v_1, \ldots, v_m|. \tag{2.1}$$

We may also define a norm $\| \cdot \|_\infty$ on Λ^m by

$$\Big\| \sum_{1 \leq i_1 < \cdots < i_m \leq m} \lambda_{i_1 \ldots i_m} (e_{i_1} \wedge \cdots \wedge e_{i_m}) \Big\|_\infty = \max |\lambda_{i_1 \ldots i_m}|. \tag{2.2}$$

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear there is an induced linear mapping $\widetilde{T} : \Lambda^m \to \Lambda^m$ given by

$$\widetilde{T}(v_1 \wedge \cdots \wedge v_m) = T(v_1) \wedge \cdots \wedge T(v_m).$$

The norms on Λ^m induce norms on the space of linear mappings $\mathcal{L}(\Lambda^m, \Lambda^m)$ in the usual way by

$$\|\widetilde{T}\| = \sup_{w \in \Lambda^m, w \neq 0} \frac{\|\widetilde{T}w\|}{\|w\|}.$$

Then with respect to the norm (2.1)

$$\|\widetilde{T}\| = \phi^m(T)$$

and with respect to the norm (2.2)

$$\|\widetilde{T}\|_\infty = \max\{|T^{(m)}| : T^{(m)} \text{ is an } m \times m \text{ minor of } T\}.$$

Recall that the $m \times m$ minor $T^{(m)} \equiv T\binom{r_1, r_2, \ldots, r_m}{s_1, s_2, \ldots, s_m}$ of the $n \times n$ matrix T is the determinant of the $m \times m$ matrix formed by the elements of T in the rows $1 \leq r_1 < \cdots < r_m \leq n$ and columns $1 \leq s_1 < \cdots < s_m \leq n$. The space of linear mappings $\mathcal{L}(\Lambda^m, \Lambda^m)$ is of finite dimension $\binom{n}{m}^2$. Since any two norms on a finite dimensional normed space are equivalent, there are constants $C_1, C_2 > 0$ depending only on n and m such that

$$C_1 \|\widetilde{T}\|_\infty \leq \|\widetilde{T}\| \leq C_2 \|\widetilde{T}\|_\infty,$$

that is,

$$\begin{aligned} C_1 \max\{|T^{(m)}| : T^{(m)} \text{ is a } m \times m \text{ minor of } T\} &\leq \phi^m(T) \\ &\leq C_2 \max\{|T^{(m)}| : T^{(m)} \text{ is a } m \times m \text{ minor of } T\}. \end{aligned} \tag{2.3}$$

Thus we may estimate the singular value function $\phi^m(T)$ by estimating $m \times m$ minors of T.

We now prove several lemmas relating to minors of matrices. We will need some well-known inequalities.

Lemma 2.1 *Let $x_i \geq 0$, $i = 1, \ldots, m$, and $p \in \mathbb{R}$. If $p > 1$, then*

$$(x_1^p + \cdots + x_m^p) \leq (x_1 + \cdots + x_m)^p \leq m^{(p-1)}(x_1^p + \cdots + x_m^p). \tag{2.4}$$

If $0 < p \leq 1$, then

$$m^{(p-1)}(x_1^p + \cdots + x_m^p) \leq (x_1 + \cdots + x_m)^p \leq (x_1^p + \cdots + x_m^p). \tag{2.5}$$

We first look at the expansion of $m \times m$ minors of the product of k matrices $A = A_1 A_2 \cdots A_k$, where

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ a_{21}^i & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & & \vdots \\ a_{n1}^i & a_{n2}^i & \cdots & a_{nn}^i \end{pmatrix}, \qquad i = 1, \ldots, k.$$

Lemma 2.2 *For $1 \leq m \leq n$, the $m \times m$ minors of $A = A_1 A_2 \cdots A_k$ have formal expansions in terms of the entries of the A_i of the form*

$$A\begin{pmatrix} r_1 & r_2 & \dots & r_m \\ s_1 & s_2 & \dots & s_m \end{pmatrix} = \sum_{c_1,c_2,\dots,c_k} \pm a^1_{1(c_1)} \cdots a^1_{m(c_1)} a^2_{1(c_2)} \cdots a^2_{m(c_2)} \cdots a^k_{1(c_k)} \cdots a^k_{m(c_k)} \quad (2.6)$$

such that for each $i = 1, \dots, k$, the $a^i_{1(c_i)} \cdots a^i_{m(c_i)}$ are distinct entries a^i_{rs} of A_i. In particular, for each i, $1(c_i), \dots, m(c_i)$ denote pairs (r, s) corresponding to entries in m different rows and m different columns of the i-th matrix A_i, and the sum is over all such entry combinations $(c_1, \dots, c_k)$ with appropriate sign $\pm$.

Proof. For $k = 2$, the Binet-Cauchy formula [10] gives

$$\begin{aligned} A\begin{pmatrix} r_1 & r_2 & \dots & r_m \\ s_1 & s_2 & \dots & s_m \end{pmatrix} &= \sum_{1\leq l_1<\dots<l_m\leq n} A_1\begin{pmatrix} r_1 & r_2 & \dots & r_m \\ l_1 & l_2 & \dots & l_m \end{pmatrix} A_2\begin{pmatrix} l_1 & l_2 & \dots & l_m \\ s_1 & s_2 & \dots & s_m \end{pmatrix} \\ &= \sum_{1\leq l_1<\dots<l_m\leq n} \sum_{p,q} (-1)^{t(p)+t(q)} a^1_{r_1 j^1_1} \cdots a^1_{r_m j^1_m} a^2_{l_1 j^2_1} \cdots a^2_{l_m j^2_m} \quad (2.7) \end{aligned}$$

where p is the permutation $\binom{l_1 \dots l_m}{j^1_1 \dots j^1_m}$ and q is the permutation $\binom{s_1 \dots s_m}{j^2_1 \dots j^2_m}$, and $t(p)$ and $t(q)$ are the signs of p and q respectively. Thus $a^i_{r_1 j^i_1}, \dots, a^i_{r_m j^i_m}$ are from different rows and different columns of $A_i, i = 1, 2$. Writing the indices of each element in the summands as $1(c_1), \dots, m(c_1)$, $1(c_2), \dots, m(c_2)$, this is of the form

$$A\begin{pmatrix} r_1 & r_2 & \dots & r_m \\ s_1 & s_2 & \dots & s_m \end{pmatrix} = \sum_{c_1,c_2} \pm a^1_{1(c_1)} \cdots a^1_{m(c_1)} a^2_{1(c_2)} \cdots a^2_{m(c_2)}.$$

Using this argument and the formula (2.7) inductively, the result follows. ∎

We now consider upper triangular matrices. For $i = 1, \dots, k$, let

$$U_i = \begin{pmatrix} u^i_1 & u^i_{12} & \cdots & u^i_{1n} \\ 0 & u^i_2 & \cdots & u^i_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u^i_n \end{pmatrix} \quad (2.8)$$

be an $n \times n$ upper triangular matrix. (Note that here, and subsequently, we abbreviate diagonal entries $u^i_{jj} = u^i_j$.) We consider the product

$$U_1 U_2 \cdots U_k = U \equiv \begin{pmatrix} u_1 & u_{12} & \cdots & u_{1n} \\ 0 & u_2 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_n \end{pmatrix}. \quad (2.9)$$

We note that

$$u_{rs} = \sum_{r\leq r_1\leq\dots\leq r_{k-1}\leq s} u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}, \qquad 1 \leq r, s \leq n, \quad (2.10)$$

since all the other products are 0.

Lemma 2.3 *With notation as in (2.8) and (2.9), let $U_1, \ldots, U_k$ be upper triangular matrices and $U = U_1 U_2 \cdots U_k$. Then*

(i) If $r > s$, $u_{rs} = 0$,

(ii) If $r = s$, $u_{rs} \equiv u_r = u_r^1 u_r^2 \cdots u_r^k$,

(iii) If $r < s$, then the sum (2.10) for u_{rs} has at most $k^{s-r} \le k^{n-1}$ non-zero terms. Moreover, each non-zero summand $u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}$ has at most $n-1$ non-diagonal terms in the product, i.e. terms with $r \neq r_1$, or $r_i \neq r_{i+1}$ or $r_{k-1} \neq s$.

Proof. (i) and (ii) are obvious.

(iii) For $r < s$, the number of terms in the sum (2.10) for u_{rs} is given by the r-th row s-th column entry $e_{rs}(k)$ of the $n \times n$ matrix

$$E(k) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}^k .$$

We use induction to prove

$$e_{rs}(k) = \binom{(k-1)+(s-r)}{k-1} \qquad r \le s, \quad k \ge 1. \tag{2.11}$$

Since $\binom{k-1}{k-1} = \binom{k}{k}$ and $\binom{(k-1)+t}{k-1} = \binom{k+t}{k} - \binom{k+t-1}{k}$, for $t = 1, 2, \ldots$, it follows that

$$\binom{k-1}{k-1} + \binom{(k-1)+1}{k-1} + \cdots + \binom{(k-1)+t}{k-1} = \binom{k+t}{k}.$$

Assuming (2.11) inductively, we have

$$\begin{aligned} e_{rs}(k+1) &= e_{rr}(k)e_{rs}(1) + e_{rr+1}(k)e_{r+1s}(1) + \cdots + e_{rs}(k)e_{ss}(1) \\ &= \binom{k-1}{k-1} + \binom{(k-1)+1}{k-1} + \cdots + \binom{(k-1)+s-r}{k-1} \\ &= \binom{k+s-r}{k}. \end{aligned}$$

Therefore $e_{rs}(k) = \binom{(k-1)+s-r}{k-1}$, for all $1 \le r \le s \le n$ and all k.

Since

$$\binom{(k-1)+t}{k-1} = \left(1 + \frac{k-1}{t}\right)\left(1 + \frac{k-1}{t-1}\right) \cdots \left(1 + \frac{k-1}{1}\right) \le k^t,$$

it follows that

$$e_{rs}(k) = \binom{(k-1)+s-r}{k-1} \le k^{s-r} \le k^{n-1}.$$

Finally, each non-zero summand $u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}$ of (2.10) has $r \le r_1 \le r_2 \le \cdots \le r_{k-1} \le s$, so we must have equality, corresponding to diagonal terms, at all but at most $s - r \le n-1$ steps. ∎

We now extend the estimate of Lemma 2.3 to minors.

Lemma 2.4 *Let $U_1, \ldots, U_k$ and U be upper triangular matrices as in (2.8) and (2.9). Then each $m \times m$ minor of U has an expansion of the form*

$$U\begin{pmatrix} r_1 & r_2 & \ldots & r_m \\ s_1 & s_2 & \ldots & s_m \end{pmatrix} = \sum_{c_1,c_2,\ldots,c_k} \pm u^1_{1(c_1)} u^2_{1(c_2)} \cdots u^k_{1(c_k)} \cdots u^1_{m(c_1)} u^2_{m(c_2)} \cdots u^k_{m(c_k)}, \quad (2.12)$$

where $1(c_i), \ldots, m(c_i)$ are as in Lemma 2.2 and

(i) there are at most $m!k^{m(n-1)}$ terms in the sum which are non-zero,

(ii) each summand contains at most $(n-1)^m$ non-diagonal elements in the product.

Proof.

(i) We may write a typical minor of U as

$$U^{(m)} \equiv U\begin{pmatrix} r_1 \ldots r_m \\ s_1 \ldots s_m \end{pmatrix} = \sum_p (-1)^{t(p)} u_{r_1 l_1} u_{r_2 l_2} \cdots u_{r_m l_m}, \quad (2.13)$$

where p is the permutation $\binom{s_1,\ldots,s_m}{l_1,\ldots,l_m}$, and $t(p)$ is the sign of p. According to Lemma 2.3, each entry u_{rs} of U is the sum of at most k^{n-1} non-zero terms of the form (2.10), so each $u_{r_1 l_1} u_{r_2 l_2} \cdots u_{r_m l_m}$ in (2.13) is the sum of at most $k^{m(n-1)}$ such terms. Counting the number of permutations p, the expansion of the minor $U^{(m)}$ in (2.6) contains at most $m!k^{m(n-1)}$ non-zero summands.

(ii) By Lemma 2.3, we know that each non-zero summand of u_{rs} in (2.10) involves at most $n-1$ non-diagonal terms in the product, hence by (2.13) each term in the sum (2.12) has at most $(n-1)^m$ non-diagonal entries.■

We now fix a set $\{T_1, \ldots, T_N\}$ of contracting upper triangular matrices

$$T_i = \begin{pmatrix} t^i_1 & t^i_{12} & \cdots & t^i_{1n} \\ 0 & t^i_2 & \cdots & t^i_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t^i_n \end{pmatrix}, \quad (2.14)$$

$i = 1, \ldots, N$, which will in the next section be the linear parts of the IFS maps (1.3). For a k-term sequence of indices $\mathbf{i} = (i_1, i_2, \ldots, i_k)$, $i_j = 1, \ldots, N$, we denote the product of matrices with these indices by

$$T_{i_1} T_{i_2} \cdots T_{i_k} \equiv T_{\mathbf{i}} = \begin{pmatrix} t^{\mathbf{i}}_1 & t^{\mathbf{i}}_{12} & \cdots & t^{\mathbf{i}}_{1n} \\ 0 & t^{\mathbf{i}}_2 & \cdots & t^{\mathbf{i}}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{\mathbf{i}}_n \end{pmatrix}. \quad (2.15)$$

Then

$$t^{\mathbf{i}}_{rs} = \sum_{r \le r_1 \le \cdots \le r_{k-1} \le s} t^{i_1}_{r r_1} t^{i_2}_{r_1 r_2} \cdots t^{i_k}_{r_{k-1} s} \qquad 1 \le r, s \le n, \quad (2.16)$$

so in particular $t^{\mathbf{i}}_j = t^{i_1}_j t^{i_2}_j \cdots t^{i_k}_j$. For notational convenience, we make the convention that $\{j_1, \ldots, j_m\}$ indicates that $\{j_1, \ldots, j_m\}$ is a set of m *distinct* integers from $\{1, \ldots, n\}$.

Theorem 2.5 *Let $\{T_1,\ldots,T_N\}$ be contracting upper triangular matrices as above. Then for $0<s\leq n$*

$$\lim_{k\to\infty}\Big[\sum_{|\mathbf{i}|=k}\phi^s(T_{\mathbf{i}})\Big]^{1/k} =$$
$$\max_{\substack{\{j_1,\ldots,j_{m-1}\}\\ \{j'_1,\ldots,j'_m\}}}\big\{|t^1_{j_1}\cdots t^1_{j_{m-1}}|^{m-s}|t^1_{j'_1}\cdots t^1_{j'_m}|^{s-m+1}+\cdots+|t^N_{j_1}\cdots t^N_{j_{m-1}}|^{m-s}|t^N_{j'_1}\cdots t^N_{j'_m}|^{s-m+1}\big\}, \tag{2.17}$$

where m is the integer such that $m-1<s\leq m$ and the maximum is over all sets $\{j_1,\ldots,j_{m-1}\}$ and $\{j'_1,\ldots,j'_m\}$ of $m-1$, respectively m, distinct integers from $\{1,\ldots,n\}$.

Proof. We first note that the singular value function $\phi^s(T)$ of any matrix T can be expressed in terms of $\phi^{m-1}(T)$ and $\phi^m(T)$ where m is the integer such that $m-1<s\leq m$. To see this, let $\alpha_1,\ldots,\alpha_n$ be the singular values of T. Then

$$\begin{aligned}\phi^s(T) &= \alpha_1\alpha_2\cdots\alpha_{m-1}\alpha_m^{s-m+1}=(\alpha_1\alpha_2\cdots\alpha_{m-1})^{m-s}(\alpha_1\alpha_2\cdots\alpha_{m-1}\alpha_m)^{s-m+1}\\ &= (\phi^{m-1}(T))^{m-s}(\phi^m(T))^{s-m+1}.\end{aligned} \tag{2.18}$$

The proof is in two parts, to show that the right side of the equation (2.17) is a lower bound for the limit, and to show it is an upper bound.

(i) Lower bound. If m is an integer we have, by (2.3),

$$\phi^m(T_{\mathbf{i}})\geq C\max\{|T_{\mathbf{i}}^{(m)}| : T_{\mathbf{i}}^{(m)} \text{ is an } m\times m \text{ minor of } T_{\mathbf{i}}\}.$$

The maximum $m\times m$ minor of $T_{\mathbf{i}}$ is at least the largest product of m distinct diagonal elements of $T_{\mathbf{i}}$, since such products are themselves minors of triangular matrices. Thus there is an number C independent of $\mathbf{i}$ such that, for all distinct $\{j_1,\ldots,j_{m-1}\}$ and all distinct $\{j'_1,\ldots,j'_m\}$,

$$\phi^{m-1}(T_{\mathbf{i}})\geq C|t_{j_1}t_{j_2}\cdots t_{j_{m-1}}| \quad\text{and}\quad \phi^m(T_{\mathbf{i}})\geq C|t_{j'_1}t_{j'_2}\cdots t_{j'_m}|.$$

Thus by (2.18), if m is the integer such that $m-1<s\leq m$ then for all $\{j_1,\ldots,j_m\}$ and $\{j'_1,\ldots,j'_m\}$

$$\begin{aligned}\sum_{|\mathbf{i}|=k}\phi^s(T_{\mathbf{i}}) &\geq C\sum_{|\mathbf{i}|=k}|t^{\mathbf{i}}_{j_1}\cdots t^{\mathbf{i}}_{j_{m-1}}|^{m-s}|t^{\mathbf{i}}_{j'_1}\cdots t^{\mathbf{i}}_{j'_m}|^{s-m+1}\\ &= C\sum_{|\mathbf{i}|=k}|t^{i_1}_{j_1}\cdots t^{i_1}_{j_{m-1}}|^{m-s}|t^{i_1}_{j'_1}\cdots t^{i_1}_{j'_m}|^{s-m+1}\cdots|t^{i_k}_{j_1}\cdots t^{i_k}_{j_{m-1}}|^{m-s}|t^{i_k}_{j'_1}\cdots t^{i_k}_{j'_m}|^{s-m+1}\\ &= C\left[|t^1_{j_1}\cdots t^1_{j_{m-1}}|^{m-s}|t^1_{j'_1}\cdots t^1_{j'_m}|^{s-m+1}+\cdots+|t^N_{j_1}\cdots t^1_{j_{m-1}}|^{m-s}|t^N_{j'_1}\cdots t^N_{j'_m}|^{s-m+1}\right]^k,\end{aligned}$$

using a multinomial expansion. Hence

$$\underline{\lim}_{k\to\infty}\Big[\sum_{|\mathbf{i}|=k}\phi^s(T_{\mathbf{i}})\Big]^{1/k}\geq$$
$$\max_{\substack{\{j_1,\ldots,j_{m-1}\}\\ \{j'_1,\ldots,j'_m\}}}\big\{|t^1_{j_1}\cdots t^1_{j_{m-1}}|^{m-s}|t^1_{j'_1}\cdots t^1_{j'_m}|^{s-m+1}+\cdots+|t^N_{j_1}\cdots t^N_{j_{m-1}}|^{m-s}|t^N_{j'_1}\cdots t^N_{j'_m}|^{s-m+1}\big\}. \tag{2.19}$$

(ii) Upper bound. Let m be the integer such that $m-1 < s \le m$. From (2.18)

$$\sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}) = \sum_{|\mathbf{i}|=k} \phi^{m-1}(T_{\mathbf{i}})^{m-s} \phi^m(T_{\mathbf{i}})^{s-m+1}.$$

Using (2.3), with $T_{\mathbf{i}}^{(m)}$ denoting a typical $m \times m$ minor of $T_{\mathbf{i}}$, we have

$$\begin{aligned}
&\sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}) \le C \sum_{|\mathbf{i}|=k} \max\Big\{|T_{\mathbf{i}}^{(m-1)}|\Big\}^{m-s} \max\Big\{|T_{\mathbf{i}}^{(m)}|\Big\}^{s-m+1} \\
&= C \sum_{|\mathbf{i}|=k} \max_{\left\{ {r_1,\dots,r_{m-1} \atop s_1,\dots,s_{m-1}} \right\}} \left| T_{\mathbf{i}} \begin{pmatrix} r_1 & \dots & r_{m-1} \\ s_1 & \dots & s_{m-1} \end{pmatrix} \right|^{m-s} \max_{\left\{ {r'_1,\dots,r'_m \atop s'_1,\dots,s'_m} \right\}} \left| T_{\mathbf{i}} \begin{pmatrix} r'_1 & \dots & r'_m \\ s'_1 & \dots & s'_m \end{pmatrix} \right|^{s-m+1} \\
&\le C \binom{n}{m}^2 \binom{n}{m-1}^2 \max_{\left\{ {r_1,\dots,r_{m-1} \atop s_1,\dots,s_{m-1}} \right\}} \max_{\left\{ {r'_1,\dots,r'_m \atop s'_1,\dots,s'_m} \right\}} \sum_{|\mathbf{i}|=k} \left| T_{\mathbf{i}}^{(m-1)} \right|^{m-s} \left| T_{\mathbf{i}}^{(m)} \right|^{s-m+1}, \qquad (2.20)
\end{aligned}$$

where C is a positive constant independent of k.

For each $\mathbf{i} = (i_1, \dots, i_k)$, we may express

$$T_{\mathbf{i}}^{(m)} \equiv T_{\mathbf{i}} \begin{pmatrix} r_1 \dots r_m \\ s_1 \dots s_m \end{pmatrix} = \sum_{c_1, c_2, \dots, c_k} \pm t^{i_1}_{1(c_1)} \cdots t^{i_1}_{m(c_1)} t^{i_2}_{1(c_2)} \cdots t^{i_2}_{m(c_2)} \cdots t^{i_k}_{1(c_k)} \cdots t^{i_k}_{m(c_k)}.$$

where the sum is over matrix entries $1(c_1), \dots, m(c_1), \dots, 1(c_k), \dots, m(c_k)$ as in Lemma 2.2. Then

$$\begin{aligned}
\left| T_{\mathbf{i}}^{(m-1)} \right|^{m-s} \left| T_{\mathbf{i}}^{(m)} \right|^{s-m+1} \le\; & \Big(\sum_{c_1,\dots,c_k} |t^{i_1}_{1(c_1)} \cdots t^{i_1}_{m-1(c_1)}| \cdots |t^{i_k}_{1(c_k)} \cdots t^{i_k}_{m-1(c_k)}| \Big)^{m-s} \\
& \times \Big(\sum_{c'_1,\dots,c'_k} |t^{i_1}_{1(c'_1)} \cdots t^{i_1}_{m(c'_1)}| \cdots |t^{i_k}_{1(c'_k)} \cdots t^{i_k}_{m(c'_k)}| \Big)^{s-m+1} \\
\le \sum_{c_1,\dots,c_k} \sum_{c'_1,\dots,c'_k} & |t^{i_1}_{1(c_1)} \cdots t^{i_1}_{m-1(c_1)}|^{m-s} |t^{i_1}_{1(c'_1)} \cdots t^{i_1}_{m(c'_1)}|^{s-m+1} \times \cdots \\
& \times |t^{i_k}_{1(c_k)} \cdots t^{i_k}_{m-1(c_k)}|^{m-s} |t^{i_k}_{1(c'_k)} \cdots t^{i_k}_{m(c'_k)}|^{s-m+1},
\end{aligned}$$

using inequality (2.5). Summing over indices $\mathbf{i}$ and factorising in the natural way,

$$\sum_{|\mathbf{i}|=k} \left|T_{\mathbf{i}}^{(m-1)}\right|^{m-s} \left|T_{\mathbf{i}}^{(m)}\right|^{s-m+1}$$

$$\begin{aligned}
&\leq \sum_{|\mathbf{i}|=k} \sum_{\substack{c_1,\ldots,c_k \\ c'_1,\ldots,c'_k}} |t^{i_1}_{1(c_1)} \cdots t^{i_1}_{m-1(c_1)}|^{m-s} |t^{i_1}_{1(c'_1)} \cdots t^{i_1}_{m(c'_1)}|^{s-m+1} \times \cdots \\
&\qquad \times |t^{i_k}_{1(c_k)} \cdots t^{i_k}_{m-1(c_k)}|^{m-s} |t^{i_k}_{1(c'_k)} \cdots t^{i_k}_{m(c'_k)}|^{s-m+1} \\
&= \sum_{\substack{c_1,\ldots,c_k \\ c'_1,\ldots,c'_k}} \Big(|t^{1}_{1(c_1)} \cdots t^{1}_{m-1(c_1)}|^{m-s} |t^{1}_{1(c'_1)} \cdots t^{1}_{m(c'_1)}|^{s-m+1} + \cdots \\
&\qquad + |t^{N}_{1(c_1)} \cdots t^{N}_{m-1(c_1)}|^{m-s} |t^{N}_{1(c'_1)} \cdots t^{N}_{m(c'_1)}|^{s-m+1} \Big) \times \cdots \\
&\qquad \times \Big(t^{1}_{1(c_k)} \cdots t^{1}_{m-1(c_k)}|^{m-s} |t^{1}_{1(c'_k)} \cdots t^{1}_{m(c'_k)}|^{s-m+1} + \cdots \\
&\qquad + |t^{N}_{1(c_k)} \cdots t^{N}_{m-1(c_k)}|^{m-s} |t^{N}_{1(c'_k)} \cdots t^{N}_{m(c'_k)}|^{s-m+1} \Big). \qquad (2.21)
\end{aligned}$$

Since we are working with products of upper triangular matrices, Lemma 2.4 implies that each non-zero term of the sum (2.21) has at most $2(n-1)^m \equiv b$ of the indices $1(c_1), \ldots, (m-1)(c_1), \ldots, 1(c_k), \ldots, (m-1)(c_k), 1(c'_1), \ldots, m(c'_1), \ldots, 1(c'_k), \ldots, m(c'_k)$ that are non-diagonal. Thus, for each set of indices $(c_1, \ldots, c_k, c'_1, \ldots, c'_k)$, we have at least $k-b$ of these indices such that $1(c_r), \ldots, (m-1)(c_r), 1(c'_r), \ldots, m(c'_r)$ are all diagonal entries. For such c_r and c'_r

$$\begin{aligned}
&|t^{1}_{1(c_r)} \cdots t^{1}_{m-1(c_r)}|^{m-s} |t^{1}_{1(c'_r)} \cdots t^{1}_{m(c'_r)}|^{s-m+1} + \cdots + |t^{N}_{1(c_r)} \cdots t^{N}_{m-1(c_r)}|^{m-s} |t^{N}_{1(c'_r)} \cdots t^{N}_{m(c'_r)}|^{s-m+1} \\
&\leq \max_{\substack{\{j_1,\ldots,j_{m-1}\} \\ \{j'_1,\ldots,j'_m\}}} \big\{ |t^{1}_{j_1} \cdots t^{1}_{j_{m-1}}|^{m-s} |t^{1}_{j'_1} \cdots t^{1}_{j'_m}|^{s-m+1} + \cdots + |t^{N}_{j_1} \cdots t^{N}_{j_{m-1}}|^{m-s} |t^{N}_{j'_1} \cdots t^{N}_{j'_m}|^{s-m+1} \big\} \\
&\qquad \equiv P_s,
\end{aligned}$$

say. Hence from (2.21)

$$\sum_{|\mathbf{i}|=k} \left|T_{\mathbf{i}}^{(m-1)}\right|^{m-s} \left|T_{\mathbf{i}}^{(m)}\right|^{s-m+1} \leq \Big[\sum_{\substack{c_1,\ldots,c_k \\ c'_1,\ldots,c'_k}} (P_s)^{(k-b)} \Big] M^b \leq C' k^q M^b (P_s)^{(k-b)},$$

where M^b bounds the terms in the expansion with non-diagonal elements, and we use Lemma 2.4(i) to bound the number of c_r and c'_r corresponding to non-zero terms, so that $C' = (m-1)!m!$ and $q = (2m-1)(n-1)$.

Combining with (2.20), we get

$$\sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}) \leq C'' k^q P_s^{(k-b)},$$

where $C'' = C\binom{n}{m-1}^2 \binom{n}{m}^2 C' M^b$ is independent of k, so

$$\overline{\lim_{k\to\infty}} \Big[\sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}) \Big]^{1/k} \leq P_s. \qquad (2.22)$$

Combining with (2.19), the theorem follows.∎

Note that the upper bound part of the above proof is complicated by having to work throughout with sums over all $\mathbf{i}$ such that $|\mathbf{i}| = k$. Unfortunately it is not enough to estimate $\phi^s(T_\mathbf{i})$ for each individual $\mathbf{i}$ and then sum.

Also note that for the indices that give the maximum in (2.17), $\{j_1, \ldots, j_{m-1}\}$ is not necessarily a subset of $\{j'_1, \ldots, j'_m\}$ although this is often the case.

Let $T_1, \ldots, T_N$ be contracting matrices and let $T_\mathbf{i}$ be the product (2.15). Recall that singular value functions are submultiplicative, i.e.

$$\phi^s(TU) \le \phi^s(T)\phi^s(U),$$

see [3]. Assuming that the T_i are non-singular contractions with singular values satisfying $1 > a \ge \alpha_1^i \ge \cdots \ge \alpha_n^i \ge b > 0$, for $1 \le i \le N$, then for $\mathbf{i} = (i_1, \ldots, i_k)$ and all $j = 1, \ldots, n$,

$$b^{|\mathbf{i}|} \le \alpha_j(T_\mathbf{i}) \le a^{|\mathbf{i}|},$$

so

$$\phi^s(T_\mathbf{i})b^{l|\mathbf{i}|} \le \phi^{(s+l)}(T_\mathbf{i}) \le \phi^s(T_\mathbf{i})a^{l|\mathbf{i}|}$$

giving

$$b^{kl} \le (\sum_{|\mathbf{i}|=k} \phi^{(s+l)}(T_\mathbf{i}))/(\sum_{|\mathbf{i}|=k} \phi^s(T_\mathbf{i})) \le a^{kl}.$$

Thus $\lim_{k\to\infty}[\sum_{|\mathbf{i}|=k} \phi^s(T_\mathbf{i})]^{1/k}$ is continuous and strictly decreasing in s; when $s = 0$ it is greater than 1, and for large s it is less than 1, so there exists a unique s for which this limit equals 1. Thus, as in [3, 4], we define

$$d(T_1, \ldots, T_N) = \Big\{s : \lim_{k\to\infty} \big[\sum_{|\mathbf{i}|=k} \phi^s(T_\mathbf{i})\big]^{1/k} = 1\Big\}. \tag{2.23}$$

Corollary 2.6 *For $T_1, \ldots, T_N$ contracting non-singular upper triangular matrices,*

$$d(T_1, \ldots, T_N) = \Big\{s : \max_{\substack{\{j_1,\ldots,j_{m-1}\} \\ \{j'_1,\ldots,j'_m\} \\ m-1<s\le m}} \big\{|t^1_{j_1} \cdots t^1_{j_{m-1}}|^{m-s}|t^1_{j'_1} \cdots t^1_{j'_m}|^{s-m+1} + \cdots + |t^N_{j_1} \cdots t^N_{j_{m-1}}|^{m-s}|t^N_{j'_1} \cdots t^N_{j'_m}|^{s-m+1}\big\} = 1\Big\}, \tag{2.24}$$

where $t^i_1, \ldots, t^i_n$ are the diagonal entries of T_i, provided this number is no greater than n. An identical result holds if $T_1, \ldots, T_N$ are lower triangular.

Proof. This is immediate from Theorem 2.5. ∎

As a consequence of Corollary 2.6, once the appropriate integer m is ascertained $d(T_1, \ldots, T_N)$ may be found by solving $\binom{n}{m-1}\binom{n}{m}$ equations corresponding to different choices of $\{j_1, \ldots, j_{m-1}\}$ and $\{j'_1, \ldots, j'_m\}$.

3 Self-affine fractals

Corollary 2.6 now gives a convenient formula for the dimensions of certain self-affine fractals.

Corollary 3.1 *Let $S_i = T_i + v_i \quad (i = 1, \ldots, N)$ be an IFS of affine contractions where T_i are upper triangular matrices, and let F be the self-affine attractor. Suppose that*

(i) the IFS satisfies the open set condition (1.2) *for a connected open set D, and*

(ii) for some $c > 0$ the projection of F onto every $(n-1)$-dimensional subspace has $(n-1)$-dimensional Lebesgue measure at least c. (In $\mathbb{R}^2$ this follows if F has a connected component that is not contained in any straight line).

Then the box-counting dimension of F is given by

$$\dim_B F = d(T_1, \ldots, T_N),$$

where $d(T_1, \ldots, T_N)$ is given by (2.24).

Proof. Under the conditions stated, the box dimension of F is $d(T_1, \ldots, T_N)$ as in (2.23), see [4, Proposition 4, Corollary 5]. (The open set condition ensures that $d(T_1, \ldots, T_N) \leq n$.) The result follows from Corollary 2.6. ∎

Corollary 3.2 *Let $S_i = T_i + v_i \quad (i = 1, \ldots, N)$ be an IFS of affine contractions where T_i are upper triangular matrices, and let F be the attractor. Suppose that $\|T_i\| < \frac{1}{2}$ for all i. Then for almost all $(v_1, \ldots, v_N) \in \mathbb{R}^{nN}$ the Hausdorff and box dimensions of F are given by*

$$\dim_H F = \dim_B F = \min\{d(T_1, \ldots, T_N), n\},$$

where $d(T_1, \ldots, T_N)$ is given by (2.24).

Proof. Under the stated conditions, the box and Hausdorff dimensions are given by $d(T_1, \ldots, T_N)$ for almost all $(v_1, \ldots, v_N)$, see [3] for $\|T_i\| < \frac{1}{3}$ and [13] for $\|T_i\| < \frac{1}{2}$, so again the conclusion follows from Corollary 2.6. ∎

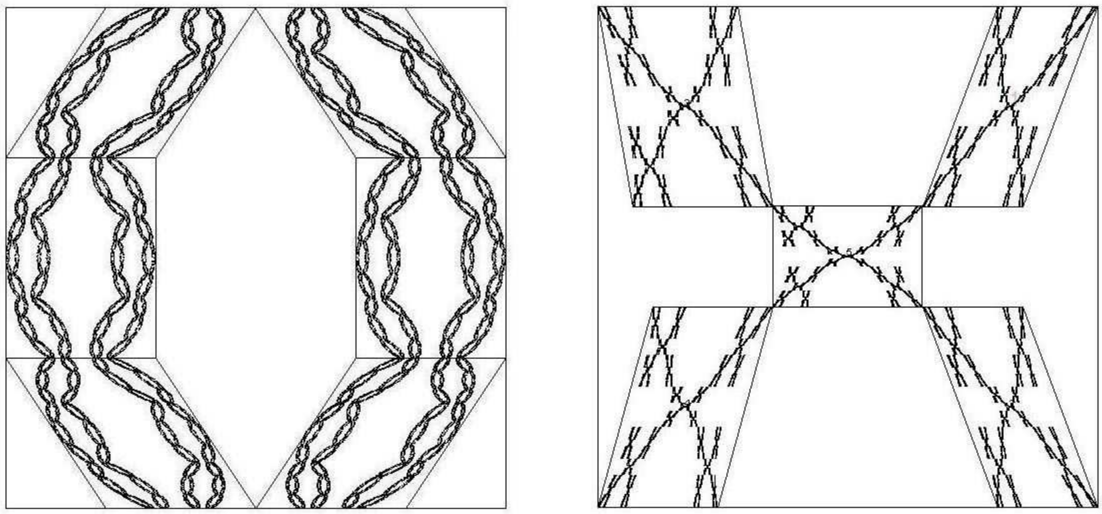
Note that there are usually specific values of $(v_1, \ldots, v_N)$ for which the conclusion of Corollary 3.2 fails, for example $S_1(x, y) = (\frac{2}{5}x, \frac{1}{3}y)$ and $S_2(x, y) = (\frac{2}{5}, \frac{1}{3}y) + (0, \frac{2}{3})$ has a Cantor set in the y-axis of dimension $\log 2/\log 3$ as attractor although $d(T_1, T_2) = \log 2/\log \frac{5}{2}$, see [6].

Finally, we always get an upper bound for the dimensions. Note that the box dimension of a set need not exist in general, so we refer to the lower and upper box dimensions $\underline{\dim}_B, \overline{\dim}_B$, see ([6]).

Corollary 3.3 *Let $S_i = T_i + v_i \quad (i = 1, \ldots, N)$ be an IFS of affine contractions where T_i are upper triangular matrices, and let F be the attractor. Then*

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq \min\{d(T_1, \ldots, T_N), n\},$$

where $d(T_1, \ldots, T_N)$ is given by (2.24).



(a) (b)

Figure 1: Self-affine fractals with box-dimensions (a) 1.576 and (b) 1.407.

Proof. The upper bound (2.23) for the dimensions is given in [3], so the conclusion follows from Corollary 2.6. ■

Note that various other conditions that ensure that the dimension of the attractor is given by $\min\{d(T_1,\dots,T_N),n\}$ are presented in [1, 8, 9, 11, 13].

Here are some illustrations of the above results.

Example 1. The two examples shown in Figures 1(a) and 1(b) satisfy the conditions of Corollary 3.1. In each case the defining affine transformations are the affine bijections from the bounding square to each of the parallelograms indicated, with horizontal lines mapped onto horizontal lines without change of orietation. Both examples are easily seen to have a connected attractor not contained in a line segment, so by Corollary 3.1 the box dimesion of the attractors are given by (2.24). Thus in Example 1(a) the box dimension is $\max\{s_1,s_2\}$, where s_1 and s_2 satisfy

$$4\cdot\tfrac{3}{10}\cdot(\tfrac{3}{10})^{s_1-1}+2\cdot\tfrac{3}{10}\cdot(\tfrac{4}{10})^{s_1-1}=1,$$

$$4\cdot\tfrac{3}{10}\cdot(\tfrac{3}{10})^{s_2-1}+2\cdot\tfrac{4}{10}\cdot(\tfrac{3}{10})^{s_2-1}=1,$$

giving $s_1=1.533, s_2=1.576$, so $\dim_B=1.576$.

For Example 1(b) (2.24) reduces to the equations

$$\tfrac{6}{25}\cdot(\tfrac{4}{10})^{s_1-1}+2\cdot\tfrac{2}{10}\cdot(\tfrac{4}{10})^{s_1-1}+\tfrac{3}{10}\cdot(\tfrac{2}{10})^{s_1-1}+\cdot\tfrac{7}{25}\cdot(\tfrac{4}{10})^{s_1-1}=1,$$

$$\tfrac{4}{10}\cdot(\tfrac{6}{25})^{s_2-1}+2\cdot\tfrac{4}{10}\cdot(\tfrac{2}{10})^{s_2-1}+\tfrac{2}{10}\cdot(\tfrac{3}{10})^{s_2-1}+\cdot\tfrac{4}{10}\cdot(\tfrac{7}{25})^{s_2-1}=1,$$

which have solutions $s_1=1.184$, $s_2=1.407$, so $\dim_B=1.407$.

Example 2. This example illustrates Corollary 3.2 Define $S_1,\dots,S_7:\mathbb{R}^2\to\mathbb{R}^2$ by

$$S_1(x,y)=(\tfrac{3}{10}x+\tfrac{1}{20}y,\tfrac{1}{5}y)+v_1,\qquad S_2(x,y)=(\tfrac{2}{10}x+\tfrac{1}{10}y,\tfrac{1}{5}y)+v_2,$$

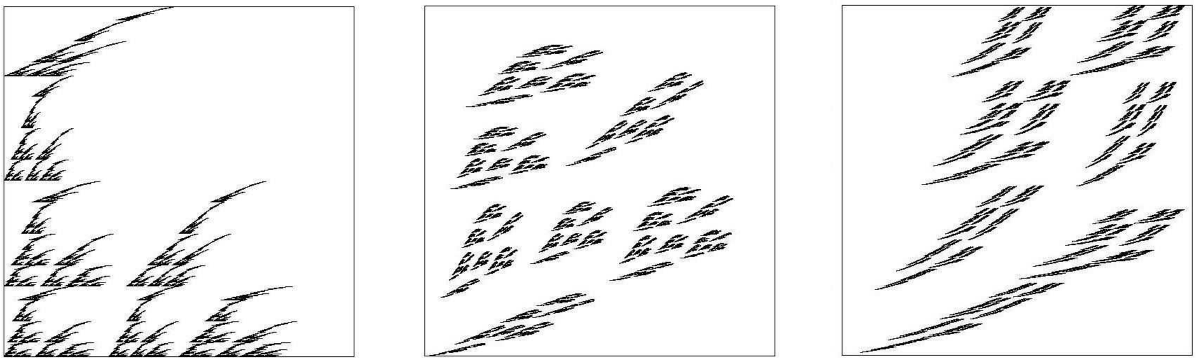


Figure 2: Three self-affine attractors with the affine maps having the same linear parts but different translation vectors. 'Almost surely' they all have box and Hausdorff dimensions 1.410.

$$S_3(x,y) = (\tfrac{7}{20}x + \tfrac{3}{20}y, \tfrac{1}{5}y) + v_3, \qquad S_4(x,y) = (\tfrac{7}{20}x + \tfrac{1}{10}y, \tfrac{3}{10}y) + v_4,$$
$$S_5(x,y) = (\tfrac{1}{5}x + \tfrac{3}{10}y, \tfrac{3}{10}y) + v_5, \qquad S_6(x,y) = (\tfrac{1}{5}x + \tfrac{1}{10}y, \tfrac{3}{10}y) + v_6,$$
$$S_7(x,y) = (\tfrac{1}{5}x + \tfrac{2}{5}y, \tfrac{1}{5}y) + v_7,$$

where $v_i \in \mathbb{R}^2$ are translation vectors. The two equations arising from (2.24) have solutions $s_1 = 1.410, s_2 = 1.383$. The conditions of Corollary 3.2 are easily verified (with $\|T_i\| < \frac{1}{2}$ with respect to the Euclidean norm), so for almost all $(v_1, \ldots, v_7) \in \mathbb{R}^{14}$, we have $\dim_H F = \dim_B F = 1.410$; see Figure 2 for three choices of v_i.

Example 3. To illustrate the procedure in higher dimensions, let $S_1, \ldots, S_5 : \mathbb{R}^3 \to \mathbb{R}^3$ be

$$\begin{aligned}
S_1(x,y,z) &= (\tfrac{1}{3}x + \tfrac{1}{10}y + \tfrac{1}{20}z, \tfrac{1}{5}y + \tfrac{1}{5}z, \tfrac{1}{6}z) + v_1,\\
S_2(x,y,z) &= (\tfrac{1}{4}x + \tfrac{1}{10}z, \tfrac{1}{5}y, \tfrac{1}{3}z) + v_2,\\
S_3(x,y,z) &= (\tfrac{1}{10}x - \tfrac{1}{10}y - \tfrac{1}{10}z, \tfrac{3}{10}y - \tfrac{1}{10}z, \tfrac{1}{4}z) + v_3,\\
S_4(x,y,z) &= (\tfrac{1}{8}x + \tfrac{1}{4}y + \tfrac{1}{10}z, \tfrac{1}{6}y + \tfrac{1}{5}z, \tfrac{1}{4}z) + v_4,\\
S_5(x,y,z) &= (\tfrac{2}{9}x - \tfrac{1}{5}y + \tfrac{1}{6}z, \tfrac{1}{7}y - \tfrac{1}{4}z, \tfrac{1}{10}z) + v_5,
\end{aligned}$$

where $v_i \in \mathbb{R}^3$ are translation vectors. This time with $1 < s < 2$ formula (2.24) necessitates solving nine equations, with solutions 1.018, 1.005, 1.018, 1.056, 1.006, 1.060, 1.016, 1.006, 1.051, so by Corollary 3.2 $\dim_H = \dim_B = 1.060$ for almost all $(v_1, \ldots, v_5) \in \mathbb{R}^{15}$.

4 Generalized dimensions of self-affine measures

We indicate how similar explicit formulae may be obtained for generalized q-dimensions of self-affine measures.

The generalised q-dimension of a finite Borel measure τ of bounded support is defined along the lines of box-counting dimension using r-mesh cubes, but reflect the power law behavior of moment sums of τ. We write $\mathcal{M}_r$ for the set of r-mesh cubes in $\mathbb{R}^n$, that is cubes of the form $[m_1 r, (m_1+1)r] \times \ldots \times [m_n r, (m_n+1)r] \subseteq \mathbb{R}^n$ where $m_1, \ldots, m_n$ are integers. For $q \in \mathbb{R}$ and $r > 0$ set

$$M_r(q) = \sum_{\mathcal{M}_r} \tau(C)^q, \tag{4.1}$$

where the sum is over the r-mesh cubes C for which $\tau(C) > 0$. We identify the power law behavior of $M_r(q)$ by defining, for $q \neq 1$, the *lower* and *upper generalized q-dimensions* of τ:

$$\underline{D}_q(\tau) = \liminf_{r\to 0} \frac{\log M_r(q)}{(q-1)\log r} \qquad \text{and} \qquad \overline{D}_q(\tau) = \limsup_{r\to 0} \frac{\log M_r(q)}{(q-1)\log r}. \tag{4.2}$$

If $\underline{D}_q(\tau) = \overline{D}_q(\tau)$, we write $D_q(\tau)$ for the common value which we refer to as the *generalized q-dimension*, see [6].

We may define a self-affine measue by weighting the affine mappings $\{S_1, S_2, \ldots, S_N\}$ of (1.3) by probabilities $\{p_1, p_2, \ldots, p_N\}$ where $p_i > 0$ and $\sum p_i = 1$. Then there exists a unique measure satisfying

$$\mu(B) = \sum_{i=1}^{N} p_i \mu(S_i^{-1}(B)), \tag{4.3}$$

for all $B \subseteq \mathbb{R}^n$. This measure is supported by the attractor F of the IFS $\{S_1, S_2, \ldots, S_N\}$, and if the strong separation condition holds, that is if the union in (1.1) is disjoint, then

$$\mu(S_{\mathbf{i}}(F)) = p_{\mathbf{i}} \equiv p_{i_1} p_{i_2} \cdots p_{i_k}.$$

Generalised q-dimensions have been calculated explicitly for self-similar measures, that is where the S_i are similarities, see [6] and references therein. For self-affine measures, the situation is more complicated, but in certain cases a formula analogous to (2.23) gives the q-dimensions. Thus for μ given by (4.3) and $q \geq 0, q \neq 1$, we define, as in [5],

$$d_q(T_1, \ldots, T_N, \mu) = \Big\{ s : \lim_{k\to\infty} \big[\sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}^{1-q}) p_{\mathbf{i}}^q \big]^{1/k} = 1 \Big\}, \tag{4.4}$$

where the limits and the unique value of $s > 0$ exist as before.

We get the following analogue of Corollary 2.6 in the upper triangular case.

Theorem 4.1 *For contracting non-singular upper triangular matrices $T_1, \ldots, T_N$,*

$$d_q(T_1, \ldots, T_N, \mu) = \Big\{ s : \max_{\substack{\{j_1,\ldots,j_{m-1}\} \\ \{j'_1,\ldots,j'_m\} \\ m-1<s\leq m}} \big\{ (|t^1_{j_1} \cdots t^1_{j_{m-1}}|^{m-s} |t^1_{j'_1} \cdots t^1_{j'_m}|^{s-m+1})^{1-q} (p_1)^q + \cdots$$
$$+ (|t^N_{j_1} \cdots t^N_{j_{m-1}}|^{m-s} |t^N_{j'_1} \cdots t^N_{j'_m}|^{s-m+1})^{1-q} (p_N)^q \big\} = 1 \Big\}, \tag{4.5}$$

where $t^i_1, \ldots, t^i_n$ are the diagonal entries of T_i, provided this number is no greater than n.

Proof. The proof is similar to that of Theorem 2.5 with Corollary 2.6. We need to replace $\sum \phi^s(T_{\mathbf{i}})$ by $\sum \phi^s(T_{\mathbf{i}})^{1-q} p_{\mathbf{i}}^q$ and terms such as $|t^{i_l}_{1(c_1)} \cdots t^{i_l}_{m-1(c_1)}|^{m-s} |t^{i_l}_{1(c'_1)} \cdots t^{i_l}_{m(c'_1)}|^{s-m+1}$ by $(|t^{i_l}_{1(c_1)} \cdots t^{i_l}_{m-1(c_1)}|^{m-s} |t^{i_l}_{1(c'_1)} \cdots t^{i_l}_{m(c'_1)}|^{s-m+1})^{(1-q)} p^q_{i_l}$. The calculation proceeds in the same way but with different powers of the t^i_j and with weights p^q_i. ∎

We have the following analogue of Corollary 3.2 for generalised dimensions.

Corollary 4.2 *Let $S_i = T_i + v_i \quad (i = 1, \ldots, N)$ be an IFS of affine contractions where T_i are upper triangular matrices, and let μ be defined by* (4.3)*. Suppose that $\|T_i\| < \frac{1}{2}$ for $1 \le i \le k$. Then for $1 < q \le 2$ the generalised q-dimensions of μ are given by*

$$D_q(\mu) = \min\{d_q(T_1, \ldots, T_N, \mu), n\}.$$

for almost all $(v_1, \ldots, v_N) \in \mathbb{R}^{nN}$, where $d_q(T_1, \ldots, T_N, \mu)$ is given by (4.5).

Proof. Under the conditions stated, the box and Hausdorff dimensions are given by $d_q(T_1, \ldots, T_N, \mu)$ for almost all $(v_1, \ldots, v_N)$, see [5], so the conclusion follows from Theorem 4.1. ∎

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